SAVIN'S ε-REGULARITY THEOREM FOR LOCALLY UNIFORMLY ELLIPTIC EQUATIONS

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ABSTRACT. In this note, we give a simple and clear proof of Savin's ε-regularity theorem [Sav07].

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1. INTRODUCTION

1.1. **Background.** We consider the fully nonlinear equation

$$F(D^2u) = 0 \quad \text{in } B_1, \tag{1.1}$$

where $F : \mathcal{S}^{n\times n} \to \mathbb{R}$ is a smooth function.

Definition 1.1. (1) F is said to be **elliptic**, if $F(M+N) \geq F(M)$ for any $M, N \in \mathcal{S}^{n\times n}$ with $N \geq 0$.

(2) F is said to be **uniformly elliptic**, if there are constants $0 < \lambda \leq \Lambda$, such that $\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|$ for any $M, N \in \mathcal{S}^{n\times n}$ with $N \geq 0$.

(3) F is said to be ρ**-uniformly elliptic** or locally uniformly elliptic, if there are constants $0 < \lambda \leq \Lambda$, such that $\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|$ for any $M, N \in \mathcal{S}^{n\times n}$ with $\|M\|, \|N\| \leq \rho$ and $N \geq 0$.

Remark 1.2. Locally uniformly elliptic equation means that the equation becomes uniformly elliptic if we have a priori Hessian bound on the solution. Many important fully nonlinear equations are locally uniformly elliptic, e.g. Monge-Ampère equations, k-Hessian equations, special Lagrangian equations.

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We first briefly review the regularity theory of uniformly elliptic equation. Let u be a viscosity solution to (1.1).

- In 50s, for two dimensional case, Nirenberg proved that u is smooth.
- In 80s, due to Krylov-Safanov's Harnack inequality, one can show that $u \in C^{1,\alpha}$ for some universal α.
- If we require F is convex or concave in additional, Evans-Krylov proved that $u \in C^{2,\alpha}$, hence smooth by Schauder theory and bootstrap argument.
- Many authors weaken the convexity/concavity condition in Evans-Krylov theory. See Caffarelli-Cabré, Caffarelli-Yuan, Collins.
- For general F, one cannot expect the $C^{2,\alpha}$ regularity, due to the counterexample of Nadirashivili.

1.2. **Main theorem.** Assume F satisfies the following hypothesis:
H1) F is elliptic and ρ-uniformly elliptic;
H2) $F(0) = 0$;
H3) $\|D^2 F\| \le K$.

Theorem 1.3. *Let F satisfy H1)-H3), and let u is a viscosity solution to*

$$F(D^2 u) = 0 \quad \text{in } B_1.$$

Then there exist constants $\delta, C > 0$, depending only on $n, \rho, \lambda, \Lambda$ and K, such that if $\|u\|_{L^\infty(B_1)} \le \delta$, then $u \in C^{2,\alpha}(B_{1/2})$ with

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C.$$

2. Applications

3. Weak Harnack inequality and Hölder regularity

In this section, we establish the weak Harnack inequality for locally uniformly elliptic equation, then use it to derive the partial $C^{0,\alpha}$ estimate.

3.1. **Preliminary.** For any $a > 0$, $y \in \overline{B}_1$, we denote the quadratic polynomial

$$P_{a,y}(x) = -\frac{a}{2}|x-y|^2 + \text{const}$$

to be a concave paraboloid centered at y of opening a.

Given $V \subset \overline{B}_1$, for any $a > 0$ and $y \in V$, we can slide the paraboloid $P_{a,y}$ from below until it touches u by below at some point $x \in \overline{B}_1$. We note that such touch point must exist, since $P_{a,y}$ touches u from below at x if and only if x is the minimum point of $u(z) + \frac{a}{2}|z-y|^2$ over $\overline{B}_1$. We collect all touch points, and denote the set of touch points by $T_a(V)$. Equivalently,

$$T_a(V) = \left\{ x : \exists\, y \in V, \text{such that } u(x) + \frac{a}{2}|x-y|^2 = \inf_z \left(u(z) + \frac{a}{2}|z-y|^2 \right) \right\}.$$

The first lemma states that if u is a supersolution and the opening a is small, then the measure of the set of touch points can control the measure of the set of centers.

Lemma 3.1. *Let F satisfy H1), and let u is a supersolution of (1.1). Set $\gamma = (n-1)\frac{\Lambda}{\lambda}+2$. Assume $a \le \frac{\rho}{\gamma}$ and $V \subset \overline{B}_1$. If $T_a(V) \subset B_1$, then*

$$|T_a(V)| \ge \frac{1}{(1+\gamma)^n}|V|.$$

Proof. We only prove this Lemma for $u \in C^2$, for general case, one can use Jensen's ε-envelope approximation argument.

For any $x \in T_a(V)$, there exists a $y \in V$, such that

$$u(x) + \frac{a}{2}|x-y|^2 = \inf_{z \in \overline{B}_1}\Big(u(z) + \frac{a}{2}|z-y|^2\Big)$$

Since $x \in B_1$ is the interior minimum point, we have

$$\begin{cases} \nabla u(x) + a(x-y) = 0, \\ D^2u(x) + aI \ge 0. \end{cases}$$

The first identity means $y = x + \frac{1}{a}\nabla u(x)$. Therefore, give a touch point in $T_a(V)$, there is only one center that corresponds to it. Now, we can define a map

$$\begin{aligned} \mathcal{M} : T_a(V) &\longrightarrow V \\ x &\longmapsto y := x + \frac{1}{a}\nabla u(x). \end{aligned}$$

$\mathcal{M}$ is clearly surjective, hence from the area forluma, we get

$$|V| \le \int_{T_a(V)} |\det \mathcal{M}(x)|\,\mathrm{d}x. \tag{3.1}$$

So far, we haven't used the equation. Next, we will use the equation to estimate D^2u on $T_a(V)$.

Claim. $-aI \le D^2u \le \gamma a I$ in $T_a(V)$.

The left inequality has been proved. Suppose the right inequality fails, then there exist $x_0 \in T_a(V)$ and a direction $e \in \mathbb{S}^{n-1}$, such that

$$D^2u(x) \ge \gamma a e \otimes e - aI.$$

Note that the eigenvalues of $\gamma a e \otimes e - aI$ are $-a, \cdots, -a, (\gamma-1)a$. Since $\gamma a \le \rho$, then $\|\gamma a e \otimes e - aI\| \le \rho$. Since F is elliptic and ρ-uniformly elliptic, then

$$0 \ge F(D^2u(x_0)) \ge F(\gamma a e \otimes e - aI) \ge \lambda(\gamma-1)a - (n-1)\Lambda a > 0,$$

contradiction! Hence the claim holds.

Note that $D\mathcal{M}(x) = I + \frac{1}{a}D^2u(x)$, then $0 \le D\mathcal{M}(x) \le (1+\gamma)I$ in $T_a(V)$. Now from (3.1), we conclude that

$$|V| \le (1+\gamma)^n|T_a(V)|.$$

$\square$

For simplicity, we denote $T_a(\overline{B}_1)$ by T_a.

Corollary 3.2. *Let F satisfy H1), and let u is a supersolution of (1.1). Assume $\rho \geq \rho_0 := 8\gamma$ and $u \geq 0$. If $\inf_{B_{1/4}} u \leq 1$, then*

$$\frac{|T_8 \cap B_1|}{|B_1|} > \mu \quad or \quad \frac{|\{u \leq 2\} \cap B_1|}{|B_1|} > \mu,$$

where $\mu = \mu(n, \lambda, \Lambda) \in (0, 1)$.

Proof. We claim that $T_8(B_{1/4}) \subset B_1$, then by Lemma 3.1, we have

$$|T_8 \cap B_1| \geq |T_8(B_{1/4})| \geq \frac{1}{(1+\gamma)^n} |B_{1/4}|.$$

Note that $T_8(B_{1/4}) \subset \{u \leq 2\}$ is obvious.

It remains to prove the claim. For any $x \in T_8(B_{1/4})$, there exists $y \in B_{1/4}$, such that x is the minimum point of $u(z) + 4|z - x|^2$. We only need to rule out the possibility that it takes the minimum at ∂B_1.

For $z \in \partial B_1$, since $u \geq 0$, then we have $u(z) + 4|z - y|^2 \geq 4 \cdot (3/4)^2 = 9/4$.

On the other hand, since $\inf_{B_{1/4}} u \leq 1$, then there exists $x_1 \in B_{1/4}$, such that $u(x_1) \leq 1$, then $u(x_1) + 4|x_1 - y|^2 \leq 2$. The claim easily holds. □

In the above argument, it is important to assume the touch point belongs to the interior of the ball, unless we cannot get the information of ∇u and $D^2 u$ at the touch point.

The next lemma is from a natural observation. If there is a touch point in the interior, then we enlarge the opening of the paraboloid and perturb the center slightly, then the corresponding touch points also belong to the interior.

Lemma 3.3. *Let F satisfy H1), and let u is a nonnegative supersolution of (1.1). Then there exists a universal $M = M(n, \lambda, \Lambda) > 1$, such that if $Ma \leq \frac{\rho}{\gamma}$, $B_r(x_0) \subset B_1$, $T_a \cap B_r(x_0) \neq \varnothing$, then*

$$\frac{|T_{Ma} \cap B_r(x_0)|}{|B_r(x_0)|} \geq \mu,$$

where $\mu = \mu(n, \lambda, \Lambda) \in (0, 1)$.

Proof. Assume $x_1 \in T_a \cap B_r(x_0)$, then there exists a $y_1 \in \overline{B_1}$, such that the paraboloid

$$P_{a,y_1}(x) = -\frac{a}{2}|x - y_1|^2 + u(x_1) + \frac{a}{2}|x_1 - y_1|^2$$

touches u from below at x_1. Next, we will enlarge the opening a to Ma and perturb the center y_1 slightly, we will show that the corresponding touch points also belong to $B_r(x_0)$. We divide the proof into 3 steps.

Step 1. We claim that $\exists\, x_2 \in \overline{B_{r/2}}(x_0)$ and $C_0 = C_0(n, \lambda, \Lambda) > 0$, such that

$$u(x_2) - P_{a,y_1}(x_2) \leq C_0 a r^2.$$

To see this, we consider the barrier function

$$\psi(x) = P_{a,y_1}(a) + ar^2 \phi\left(\frac{|x - x_0|}{r}\right),$$

where

$$\phi(t) = \begin{cases} \frac{1}{p}(t^{-p} - 1) & \text{for } \frac{1}{2} \leq t \leq 1, \\ \frac{1}{p}(2^p - 1) & \text{for } t < \frac{1}{2}. \end{cases}$$

Set x_2 be the minimum point of $u - \psi$ over $\overline{B_r}(x_0)$.

Note that $F(D^2\psi) \geq 0$ in $B_{r(x_0)} \setminus \overline{B_{r/2}}(x_0)$ provided p is universal large. By the definition of viscosity solutions or comparison principle. $u-\psi$ cannot attain its minimum in $B_r(x_0) \setminus \overline{B_{r/2}}(x_0)$, that is $x_2 \notin B_r(x_0) \setminus \overline{B_{r/2}}(x_0)$.

For $z \in \partial B_r(x_0)$, we have

$$u(z) - \psi(z) = u(z) - P_{a,y_1}(z) \geq 0.$$

However,

$$u(x_1) - \psi(x_1) = u(x_1) - P_{a,y_1}(x_1) = -ar^2\phi\left(\frac{|x_1 - x_0|}{r}\right) < 0.$$

Therefore $x_2 \notin \partial B_r(x_0)$.

Finally, we have $x_2 \in \overline{B_{r/2}}(x_0)$ with $(u - \psi)(x_2) < 0$. Since $\phi \leq C_0$ for universal C_0, then

$$u(x_2) < \psi(x_2) \leq P_{a,y_1}(x_2) + C_0 ar^2.$$

Step 2. We show that $T_{Ma}(V) \subset B_r(x_0)$ with

$$V = \overline{B_{r\frac{M-1}{8M}}}\left(\frac{1}{M}y_1 + \frac{M-1}{M}x_2\right).$$

For any $\widetilde{x} \in T_{Ma}(\widetilde{y})$, there exists $\widetilde{y} \in V$, such that the paraboloid

$$P_{Ma,\widetilde{y}}(x) = -\frac{Ma}{2}|x - \widetilde{y}|^2 + u(\widetilde{x}) + \frac{Ma}{2}|\widetilde{x} - \widetilde{y}|^2$$

touches u from below at $\widetilde{x}$.

Now we have two inequalities

$$\begin{cases} P_{a,y_1}(x) \leq u(x) & \text{in } B_1, \text{ with equality at } x_1; \\ P_{Ma,\widetilde{y}}(x) \leq u(x) & \text{in } B_1, \text{ with equality at } \widetilde{x}. \end{cases}$$

We first examine the difference of this two paraboloid, note that

$$\begin{aligned} P_{Ma,\widetilde{y}}(x) - P_{a,y_1}(x) &= -\frac{Ma}{2}|x - \widetilde{y}|^2 + \frac{a}{2}|x - y_1|^2 + \text{Const} \\ &= -\frac{(M-1)a}{2}|x - y^*|^2 + R, \end{aligned}$$

where $y^* = \frac{M\widetilde{y} - y_1}{M-1}$ and R denote the remainder constant term. Since $\widetilde{y} \in V$, by the definition of V, we have $y^* \in \overline{B_{r/8}}(x_2)$.

To estimate the remainder term R, we note that

$$P_{a,y_1}(x) - \frac{(M-1)a}{2}|x - y^*|^2 + R = P_{Ma,\widetilde{y}}(x) \leq u(x) \quad \text{in } B_1,$$

then, at the point x_2, by Step 1, we get

$$\begin{aligned} R &\le u(x_2) - P_{a,y_1}(x_2) + \frac{(M-1)a}{2}|x_2 - y^*|^2 \\ &\le C_0 a r^2 + \frac{M-1}{128} a r^2 = \left(C_0 + \frac{M-1}{128}\right) a r^2. \end{aligned}$$

Our goal is to show that $\widetilde{x} \in B_r(x_0)$, thus we need to estimate $|\widetilde{x} - x_0|$. To do this, we first estimate $|\widetilde{x} - y^*|$. Since

$$\begin{aligned} 0 &\le u(\widetilde{x}) - P_{a,y_1}(\widetilde{x}) = P_{Ma,\widetilde{y}}(\widetilde{x}) - P_{a,y_1}(\widetilde{x}) \\ &= -\frac{(M-1)a}{2}|\widetilde{x} - y^*|^2 + R, \end{aligned}$$

then

$$|\widetilde{x} - y^*|^2 \le \frac{2}{(M-1)a} R \le \left(\frac{2C_0}{M-1} + \frac{1}{64}\right) r^2.$$

Hence, $|\widetilde{x} - y^*| \le \frac{r}{4}$ provided M is universal large. Finally,

$$|\widetilde{x} - x_0| \le |\widetilde{x} - y^*| + |y^* - x_2| + |x_2 - x_0| \le \frac{r}{4} + \frac{r}{8} + \frac{r}{2} < r.$$

Step 3. Conclusion. By Lemma 3.1, we have

$$|T_{Ma} \cap B_r(x_0)| \ge |T_{Ma}(V)| \ge c|V| \ge c\left(r\frac{M-1}{8M}\right)^n = \widetilde{c} r^n,$$

which implies

$$\frac{|T_{Ma} \cap B_r(x_0)|}{|B_r(x_0)|} > \mu$$

for some universal $\mu \in (0,1)$. □

We also need the following covering lemma.

Lemma 3.4. *Let $E \subset F \subset B_1$, with $E \neq \varnothing$, and let $\mu \in (0,1)$. If for any ball $B \subset B_1$ with $B \cap E \neq \varnothing$, we have $|B \cap F| > \mu|B|$, then*

$$|B_1 \setminus F| \le \left(1 - \frac{\mu}{5^n}\right)|B_1 \setminus E|.$$

Corollary 3.5. *Let F satisfy H1), and let u is a nonnegative supersolution of (1.1). Assume $\rho \ge \rho_0 := 8\gamma$. If $\inf_{B_{1/4}} u \le 1$, then*

$$|B_1 \setminus T_{8M^k}| \le C_n(1-\theta)^k \quad \textit{provided } 1 \le k \le \frac{1}{\ln M}\ln\frac{\rho}{\rho_0}.$$

where M is the constant in Lemma 3.3 and $\theta = \theta(n,\lambda,\Lambda) \in (0,1)$.

Proof. We prove this corollary by induction. For $k = 1$, the result follows from Corollary 3.2. Assume this result holds for $k-1$. Denote $E = T_{8M^{k-1}} \cap B_1$ and $F = T_{8M^k} \cap B_1$, then

$E \subset F \subset B_1$. For any ball $B = B_r(x_0) \subset B_1$ with $B \cap E \neq \varnothing$, that is $T_{8M^{k-1}} \cap B_r(x_0) \neq \varnothing$. By Lemma 3.3, we have

$$\frac{|T_{8M^k} \cap B|}{|B|} \geq \mu \quad \text{provided } 8M^k \leq \frac{\rho}{\gamma}, \text{ or equivalently } k \leq \frac{1}{\ln M} \ln \frac{\rho}{\rho_0}.$$

Hence, by Lemma 3.4, we have $|B_1 \setminus F| \leq (1 - 5^{-n}\mu)|B_1 \setminus E|$, that is

$$|B_1 \setminus T_{8M^k}| \leq \left(1 - \frac{\mu}{5^n}\right) |B_1 \setminus T_{8M^{k-1}}| \leq (1-\theta)^k,$$

provided $k \leq \dfrac{1}{\ln M} \ln \dfrac{\rho}{\rho_0}$ and if we take $\theta = 5^{-n}\mu$. □

A direct consequence of Lemma 3.5 is the following weak L^ε estimate.

Corollary 3.6 (Weak L^ε estimate)**.** *Under the hypothesis of Corollary 3.5, we have*

$$|\{u > t\} \cap B_1| < Ct^{-\varepsilon} \quad \textit{for } 0 \leq t \leq 17\frac{\rho}{\rho_0},$$

where $\varepsilon = \varepsilon(n, \lambda, \Lambda) > 0$.

Proof. Since $\inf_{B_{1/4}} u \leq 1$, then $T_{8M^k} \subset \{u \leq 1 + 16M^k\} \subset \{u \leq 17M^k\}$. For $17 < t \leq 20\frac{\rho}{\rho_0}$, there exists $k \in \mathbb{N}$, such that $17M^k \leq t < 17M^{k+1}$, that is, $k = \left[\frac{1}{\ln M} \ln \frac{t}{17}\right]$. Since $t \leq 17\frac{\rho}{\rho_0}$, then $k \leq \frac{1}{\ln M} \ln \frac{\rho}{\rho_0}$. Hence, by Lemma 3.5, we get

$$|\{u > t\} \cap B_1| \leq |\{u > 1 + 16M^k\} \cap B_1| \leq |B_1 \setminus T_{8M^k}| \leq (1-\theta)^k \leq Ct^{-\varepsilon}.$$

For $0 < t \leq 17$, the result holds clearly. □

The weak L^ε estimate implies the following weak Harnack inequality.

Theorem 3.7 (Weak Harnack)**.** *Let F satisfy H1), and let u is a nonnegative supersolution of (1.1). Assume $\rho > \rho_0$ and $\inf_{B_{1/4}} u \leq \frac{\rho}{\rho_0}$. Then there exists $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) > 0$, such that*

$$\|u_\rho\|_{L^{\varepsilon_0}(B_{1/4})} \leq C \inf_{B_{1/4}} u,$$

where $C = C(n, \lambda, \Lambda) > 0$ and

$$u_\rho(x) = \begin{cases} u(x), & \textit{if } u(x) \leq 17\rho/\rho_0, \\ 0 & \textit{if } u(x) > 17\rho/\rho_0. \end{cases}$$

Proof. Set $B := \inf_{B_{1/4}} u \leq \frac{\rho}{\rho_0}$, and $v = u/B$, then $\inf_{B_{1/4}} v = 1$. Moreover, v is a nonnegative supersolution to $\widetilde{F}(D^2 v) \leq 0$, where $\widetilde{F}(\cdot) = \frac{1}{B} F(B\cdot)$. Note that $\widetilde{F}$ is $\frac{\rho}{B}$-uniformly elliptic with same ellipticity constants $0 < \lambda \leq \Lambda$. By Corollary 3.6, we have

$$|\{v > t\} \cap B_1| < Ct^{-\varepsilon}, \quad \textit{for } 0 \leq t \leq 17\frac{\rho}{\rho_0 B}.$$

Set $v_\rho = u_\rho/B$ and $\varepsilon_0 = \varepsilon/2$, then we have

$$\begin{aligned}\int_{B_1} |v_\rho|^{\varepsilon_0} &= \varepsilon_0 \int_0^\infty t^{\varepsilon_0 - 1} |\{v_\rho > t\} \cap B_{1/4}| \,\mathrm{d}t \\ &\le \varepsilon_0 \left(\int_0^1 |B_{1/4}| \,\mathrm{d}t + \int_1^{17\frac{\rho}{\rho_0 B}} t^{\varepsilon_0 - 1} |\{v > t\} \cap B_1| \,\mathrm{d}t \right) \\ &\le \varepsilon_0 |B_{1/4}| + C \int_1^{17\frac{\rho}{\rho_0 B}} t^{\varepsilon_0 - 1 - \varepsilon} \,\mathrm{d}t \\ &\le C\end{aligned}$$

Hence, $\|v_\rho\|_{L^{\varepsilon_0}(B_{1/4})} \le C$, which means $\|u_\rho\|_{L^{\varepsilon_0}(B_{1/4})} \le CB = C \inf_{B_{1/4}} u$. □

Theorem 3.8 (Hölder regularity). *Let F satisfy H1), and let u is a viscosity solution of (1.1). Assume $\rho > 2\rho_0$ and $\|u\|_{L^\infty(B_1)} \le 1$, then*

$$\mathrm{osc}_{B_r} u \le C r^\alpha, \quad for \ \sqrt{\frac{2\rho_0}{\rho}} \le r \le 1.$$

where $C > 0, \alpha \in (0,1)$ are universal constant depending only on n, λ, Λ.

Proof. We only need to prove

$$\mathrm{osc}_{B_{4^{-k}}} u \le C(1-\kappa)^k, \quad \text{for } 0 \le k \le \frac{1}{2\ln 4} \ln \frac{\rho}{2\rho_0}.$$

where C, κ are universal. We prove it by induction. For $k = 0$, it holds obviously. Assume it holds for k, set $r_k = 4^{-k}$, $M_k = \sup_{B_{4^{-k}}} u, m_k = \inf_{B_{4^{-k}}} u$ and $\omega_k = M_k - m_k$.

WLOG, assume that $\omega_k \ge r_k^2$. Consider the rescaled function

$$v(y) = \frac{u(r_k y) - m_k}{M_k - m_k}, \quad y \in B_1,$$

then v is a viscosity solution to $\widetilde{F}(D^2 v) = 0$ with $\widetilde{F}(\cdot) = \frac{r_k^2}{\omega_k} F\left(\frac{\omega_k}{r_k^2} \cdot\right)$. Note that $\widetilde{F}$ is $\frac{r_k^2}{\omega_k}\rho$-uniformly elliptic with same ellipticity constants $0 < \lambda \le \Lambda$.

Since $0 \le v \le 1$ with $\mathrm{osc}_{B_1} v = 1$, then only one of following holds:

$$\frac{|\{v \ge 1/2\} \cap B_1|}{|B_1|} \ge \frac{1}{2} \quad or \quad \frac{|\{v \le 1/2\} \cap B_1|}{|B_1|} \ge \frac{1}{2}$$

WLOG, assume the first case holds. Note that $\|u\|_{L^\infty(B_1)} \le 1$, then $\omega_k \le 2$. Since $k \le \frac{1}{2\ln 4} \ln \frac{\rho}{2\rho_0}$, then

$$\frac{r_k^2}{\omega_k} \rho \ge \frac{4^{-k}}{2} \rho \ge \rho_0.$$

Therefore, we applying Lemma 3.6 to conclude that

$$c_0 \le \|v_{r_k^2 \rho/2}\|_{L^{\varepsilon_0}(B_{1/4})} \le C \inf_{B_{1/4}} v.$$

This implies $\inf_{B_{1/4}} v \geq \kappa$ for some universal κ, then $\frac{m_{k+1}-m_k}{\omega_k} \geq \kappa$. Hence,

$$\omega_{k+1} = M_{k+1} - m_{k+1} \leq M_k - m_{k+1} \leq (1-\kappa)\omega_k \leq (1-\kappa)^{k+1}.$$

□

By translation, we have the following corollary which is useful in our future compactness argument.

Corollary 3.9. *Let F satisfy H1), and let u is a viscosity solution of (1.1). Assume $\rho > 2\rho_0$ and $\|u\|_{L^\infty(B_1)} \leq 1$, then*

$$\operatorname{osc}_{B_r(x_0)} u \leq Cr^\alpha, \quad \textit{for any } x_0 \in B_{1/2} \textit{ and } \sqrt{\frac{2\rho_0}{\rho}} \leq r \leq \frac{1}{2}.$$

where $C > 0, \alpha \in (0,1)$ are universal constant depending only on n, λ, Λ.

Consequently,

$$|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^\alpha, \quad \textit{for any } x_1, x_2 \in B_{1/2} \textit{ and } \sqrt{\frac{2\rho_0}{\rho}} \leq |x_1 - x_2| \leq \frac{1}{2}.$$

4. Proof of Theorem 1.3

Lemma 4.1 (Improvement of flatness)**.** *Let F satisfy H1)-H3), and let u be a viscosity solution to*

$$F(D^2u) = 0 \quad \text{in } B_1.$$

There exists universal constants $\delta_0 > 0$, $\eta \in (0,1)$ and $C > 0$, which depend only on $n, \lambda, \Lambda, \rho$ and K, such that if $\|u\|_{L^\infty(B_1)} \leq \delta_0$, then there exists a quadratic polynomial P with $F(D^2P) = 0$, such that

$$|P| \leq C \sup_{B_1} |u| \quad \textit{and} \quad \sup_{B_\eta} |u - P| \leq \eta^{2+\alpha} \sup_{B_1} |u|.$$

Proof. We prove this lemma by contradiction. If this lemma fails, for universal $\eta \in (0,1)$ and $C > 0$ to be chosen later, there exist $\{F_k\}$ and $\{u_k\}$ such that

- $\{F_k\}$ satisfy H1)-H3);
- $F_k(D^2u_k) = 0$ in B_1 in viscosity sense;
- $\delta_k := \sup_{B_1} |u_k| \to 0$;
- For any k, and any quadratic polynomial P with $F_k(D^2P) = 0$ and $P \leq C\delta_k$, we have $\sup_{B_\eta} |u_k - P| > \eta^{2+\alpha}\delta_k$.

Step 1. Passing to limit.

Since $\{F_m\}$ are ρ-uniformly elliptic, and $\|DF_m\| \leq K$, then by Arzela-Ascoli theorem, up to subsequence, DF_m converges to some matrix-valued function A locally uniformly in the ρ-neighborhood of the origin in $\mathcal{S}^{n\times n}$.

Consider $v_k = \delta_k^{-1}u_k$, then v_k satisfies $\widetilde{F}_k(D^2v_k) = 0$ in the viscosity sense, where $\widetilde{F}_k(\cdot) = \delta_k^{-1}F(\delta_k\cdot)$. Note that $\widetilde{F}_k$ is $\delta_k^{-1}\rho$-uniformly elliptic. For sufficiently large k, we have $\delta_k^{-1}\rho > 2\rho_0$. Thus we can apply Corollary 3.9 for v_k to get

$$|v_k(x_1) - v_k(x_2)| \le C|x_1 - x_2|^\alpha, \quad for\ any\ x_1, x_2 \in B_{1/2}\ and\ \sqrt{\frac{2\rho_0}{\delta_k^{-1}\rho}} \le |x_1 - x_2| \le \frac{1}{2}.$$

Set $d_k = \sqrt{\frac{2\rho_0}{\delta_k^{-1}\rho}}$. Since $\delta_k \to 0$, then for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}^+$, such that $2Cd_k^\alpha < \varepsilon$ for any $k \ge k_0$. Now for any $x_1, x_2 \in B_{1/2}$, with $|x_1 - x_2| \le d_{k_0}$, we can choose $x_3 \in B_{1/2}$ such that $|x_1 - x_3| = |x_2 - x_3| = d_{k_0}$. Now for any $k \ge k_0$, we have

$$|v_k(x_1) - v_k(x_2)| \le |v_k(x_1) - v_k(x_3)| + |v_k(x_2) - v_k(x_3)| \le 2C\delta_{k_0}^\alpha < \varepsilon.$$

Therefore, $\{v_k\}$ is equicontinuous. By Arzela-Ascoli theorem, up to subsequence, $v_k \to v_0$ in $C(B_{1/2})$.

Step 2. v_0 is the viscosity solution to $A^{ij}(0)D_{ij}v_0 = 0$ in $B_{1/2}$.

For any $x_0 \in B_{1/2}$ and $\varphi \in C^2$ such that φ touches v_0 by above at x_0. Then for sufficiently large k, φ touches $v_k = \delta_k^{-1}u_k$ by above at some $x_k \in B_{1/2}$. Thus $\delta_k\varphi$ touches u_k by above at x_k. Then

$$\begin{aligned}
0 \le F_k(\delta_k D^2\varphi(x_k)) &= \int_0^{\delta_k} \frac{\mathrm{d}}{\mathrm{d}t} F_k(tD^2\varphi(x_k))\,\mathrm{d}t = \int_0^{\delta_k} F_k^{ij}(tD^2\varphi(x_k))D_{ij}\varphi(x_k)\,\mathrm{d}t \\
&= \int_0^{\delta_k} F_k^{ij}(0)D_{ij}\varphi(x_k) + \left(F_k^{ij}(tD^2\varphi(x_k)) - F_k^{ij}(0)\right) D_{ij}\varphi(x_k)\,\mathrm{d}t. \\
&\le \delta_k F_k^{ij}(0)D_{ij}\varphi(x_k) + \int_0^{\delta_k} Kt\|D^2\varphi(x_k)\|^2\,\mathrm{d}t \\
&= \delta_k \left(F_k^{ij}(0)D_{ij}\varphi(x_k) + \frac{\delta_k}{2}K\|D^2\varphi(x_k)\|^2\right)
\end{aligned}$$

Here $F_k^{ij}(M) = \frac{\partial F_k}{\partial m_{ij}}(M)$. Therefore, $F_k^{ij}(0)D_{ij}\varphi(x_k) + \frac{\delta_k}{2}K\|D^2\varphi(x_k)\|^2 \ge 0$, sending $k \to \infty$, we get $A^{ij}(0)D_{ij}\varphi(x_0) \ge 0$. This implies v_0 is a viscosity subsolution to $A^{ij}(0)D_{ij}v_0 = 0$. Similarly, we can also prove that v_0 is a supersolution.

Step 3. Derive the contradiction.

Since F_k is ρ-uniformly elliptic, then $\lambda I \le \left(F_k^{ij}(0)\right) \le \Lambda I$, passing to limit, we get $\lambda I \le \left(A^{ij}(0)\right) \le \Lambda I$. Therefore, v_0 is the viscosity solution to a linear uniformly elliptic equation with constant coefficients. Moreover, $\|v_0\|_{L^\infty(B_{1/2})} \le 1$, then $v_0 \in C^\infty(B_{1/2})$, with $\|D^k v_0\|_{L^\infty(B_{1/4})} \le C = C(n, k, \lambda, \Lambda)$. Let P be the quadratic Taylor polynomial of v_0, that is $P(x) = v_0(0) + \nabla v_0(0)\cdot x + \frac{1}{2}x^T D^2 v_0(0)x$, then $|P| \le C = C(n, \lambda, \Lambda)$ and

$$\sup_{B_\eta} |v_0 - P| \le C\eta^3 \le \frac{1}{2}\eta^{2+\alpha},$$

provided $C\eta^{1-\alpha} < 1/2$.

Note that

$$\begin{aligned} F_k(\delta_k D^2 P) &= \int_0^{\delta_k} \frac{\mathrm{d}}{\mathrm{d}t} F_k(tD^2P)\,\mathrm{d}t = \int_0^{\delta_k} F_k^{ij}(tD^2P) D_{ij}P\,\mathrm{d}t \\ &\le \delta_k F_k^{ij}(0) D_{ij}P + \frac{\delta_k^2}{2} K \|D^2 P\|^2 \\ &= o(\delta_k). \end{aligned}$$

Since F_k is ρ-uniformly elliptic, then for k large, there exists $a_k = o(\delta_k)$, such that $P_k(x) := \delta_k P(x) + a_k|x|^2$ satisfies $F_k(D^2P_k) = 0$ and $|P_k| \le C\delta_k$. Now,

$$\begin{aligned} \sup_{B_\eta} |u_k - P_k| &= \sup_{B_\eta} |u_k - \delta_k P - a_k|x|^2| \\ &\le \delta_k \left(\sup_{B_\eta} \left| \frac{u_k}{\delta_k} - P \right| + \frac{a_k}{\delta_k}\eta^2 \right) \\ &\le \delta_k \left(\sup_{B_\eta} |v_k - v_0| + \sup_{B_\eta} |v_0 - P| + o(1)\eta^2 \right) \\ &\le \delta_k \left(o(1) + \frac{1}{2}\eta^{2+\alpha} + o(1)\eta^2 \right) \\ &\le \delta_k \eta^{2+\alpha}, \end{aligned}$$

for sufficiently large k. This leads a contradiction. □

Finally, applying Caffarelli's iteration argument, we get the interior $C^{2,\alpha}$ estimate for flat solutions.

Lemma 4.2. *Under the hypothesis of Lemma 4.1, if $\|u\|_{L^\infty(B_1)} = \delta \le \delta_0$, then there exists a quadratic polynomial P with $F(D^2P) = 0$ and $|P| \le C\delta$, such that*

$$|u(x) - P(x)| \le C\delta|x|^{2+\alpha}, \quad \textit{for any } x \in B_1.$$

Proof. This proof is standard, see

Step 1. We construct a sequence quadratic polynomial $\{P_k\}_{k=0}^\infty$, such that

$$F(D^2P_k) = 0, \quad \textit{and} \quad \sup_{B_{\eta^k}} |u - P_k| \le \delta\eta^{(2+\alpha)k}.$$

We prove it by induction. For $k = 0$, $P_0 = 0$ is desired. Assume it holds for k, consider the rescaled function

$$\widetilde{u}(x) = \frac{u(\eta^k x) - P_k(\eta^k x)}{\eta^{2k}}, \ x \in B_1 \quad \textit{and} \quad \widetilde{F}(\cdot) := F(\cdot + D^2P_k).$$

Then $\|\widetilde{u}\|_{L^\infty(B_1)} \le \delta\eta^{k\alpha} \le \delta_0$, by Lemma 4.1, there exists a quadratic polynomial $\widetilde{P}_k$, such that $\widetilde{F}(D^2\widetilde{P}_k) = 0$, $|\widetilde{P}_k| \le C\sup_{B_1}|\widetilde{u}| \le C\delta\eta^{k\alpha}$ and

$$\sup_{B_\eta} |\widetilde{u} - \widetilde{P}_k| \le \eta^{2+\alpha} \sup_{B_1} |u| \le \delta\eta^{k\alpha}\eta^{2+\alpha}.$$

Transform back to u and denote $P_{k+1}(x) = P_k(x) + \eta^{2k}\widetilde{P}_k(\eta^{-k}x)$, we have

$$F(D^2 P_{k+1}) = 0 \quad \textit{and} \quad \sup_{B_{\eta^{k+1}}} |u - P_{k+1}| \leq \delta\eta^{(2+\alpha)(k+1)}.$$

Step 2. Show that $\{P_k\}$ will converge, and the limit is our desired polynomial.

By the construction in Step 1, $P_{k+1} - P_k = \eta^{2k}\widetilde{P}_k(\eta^{-k}x)$. Assume $\widetilde{P}_k(x) = a_k + b_k \cdot x + x^T C_k x$, since $|\widetilde{P}_k| \leq C\delta\eta^{k\alpha}$, we have $|a_k| + |b_k| + \|C_k\| \leq C\delta\eta^{k\alpha}$. Now

$$P_{k+1}(x) - P_k(x) = \eta^{2k}a_k + \eta^k b_k \cdot x + x^T C_k x.$$

Since $\sum \eta^{2k}|a_k|, \sum \eta^k |b_k|, \sum \|C_k\|$ converge, then $\sum_{k=0}^{\infty}(P_{k+1} - P_k)$ converges. Thus $\lim_{k\to\infty} P_k = P$ for some quadratic polynomial P. Moreover, $F(D^2P) = 0$. Since

$$P(x) = \left(\sum_{k=0}^{\infty} \eta^{2k} a_k\right) + \left(\sum_{k=0}^{\infty} \eta^k b_k\right) \cdot x + x^T \left(\sum_{k=0}^{\infty} C_k\right) x,$$

it is clearly that

$$|P| \leq C\delta, \quad \textit{and} \quad \sup_{B_{\eta^k}} |P - P_k| \leq C\delta\eta^{(2+\alpha)k}.$$

Finally, for any $x \in B_1$, there exists $k \in \mathbb{N}$, such that $\eta^{k+1} < |x| \leq \eta^k$, then

$$\begin{aligned} |u(x) - P(x)| &\leq |u(x) - P_k(x)| + |P(x) - P_k(x)| \\ &\leq \delta\eta^{(2+\alpha)k} + C\delta\eta^{(2+\alpha)k} \\ &\leq C\delta|x|^{2+\alpha}. \end{aligned}$$

□

References

[Sav07] Ovidiu Savin, *Small perturbation solutions for elliptic equations*, Comm. Partial Differential Equations **32** (2007), no. 4-6, 557–578. MR 2334822