NOTES IN ELLIPTIC PDE

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Abstract. In this note, we will collect some interested small results about elliptic PDE and geometric analysis.

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1. New proof of Evans-Krylov theorem

In this section, we present a new proof of Evans-Krylov theorem which is due to L. Caffarelli and L. Silvestre [CS10].

Theorem 1.1 (Evans-Krylov)**.** *Let F be a fully nonlinear, convex, uniformly elliptic operator with $F(0) = 0$. Assume that u is a smooth solution to $F(D^2u) = 0$ in B_1, then*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{C^{1,1}(B_1)},$$

where C depends only on n and the ellipticity constants of F.

The proof is based on two facts: Krylov-Safanov Harnack inequality and the uniform ellipticity of F.

First, we recall K-S Harnack inequality. Consider the linear uniformly elliptic operatpr $L = a_{ij}\partial_{ij}$, where $\lambda I \leq (a_{ij}) \leq \Lambda I$. K-S estimates contains two parts:

1) (Local maximum principle) Let v be a subsolution to $Lv \geq 0$ in B_1, with $0 \leq v \leq 1$, then

$$\|v\|_{L^\infty(B_{1/2})} \leq C|\{v > 0\} \cap B_{3/4}|.$$

2) (Weak Harnack/ L^ε estimate) Let v be a nonnegative supersolution to $Lv \leq 0$ with $\inf_{B_{1/2}} v = 1$, then v belongs to weak L^ε for some uniformly small ε, that is

$$|\{v > t\} \cap B_{1/4}| \leq Ct^{-\varepsilon}.$$

From the concavity of F, for any direction $e \in \mathbb{S}^{n-1}$, the second derivative u_{ee} satisfies $F_{ij}(D^2u)(u_{ee})_{ij} \leq 0$, which means that u_{ee} is a supersolution to the linearized equation. Then we can get a one-side control on D^2u.

For the other side, we need a natural observation. Since F is uniformly elliptic and $F(0) = 0$, then the positive part and the negative part of D^2u is comparable, this improve the one-side control to two-side control.

Now we proceed the proof of Theorem 1. By scaling, we may assume $\|u\|_{C^{1,1}(B_1)} = 1$, it suffices to show that the Hölder modulus of D^2u has a universal bound. For any subspace $V \subset \mathbb{R}^n$, we define

$$w(x, V) = \Delta_V u(x) = \sum_{i=1}^{k} D^2_{\varepsilon_i\varepsilon_i} u,$$

where $\{\varepsilon_1, \cdots, \varepsilon_k\}$ is an orthonormal basis of V. From the discussion above, w is a supersolution to the linearized equation. Note that the positive and negative part of D^2u can be expressed by

$$\max_V w(x, V) = \mathrm{tr} D^2u(x)^+; \qquad \min_V w(x, V) = -\mathrm{tr} D^2u(x)^-.$$

By the standard iteration argument, the Hölder estimate of w follows from the following oscillation decay lemma.

Lemma 1.2. *There exists a universal constant $\theta > 0$, such that if*

$$\min_V \inf_{x\in B_1} w(x, V) \geq -1.$$

Then,

$$\min_V \inf_{x\in B_{1/2}} w(x,V) \geq -1+\theta.$$

Proof. WLOG, by subtracting some quadratic polynomial, we may assume $D^2u(0)=0$. Let $\theta>0$ small to be chosen later. Denote $L=F_{ij}\partial_{ij}$ be the linearized operator.

We shall prove this lemma by contradiction. If there exist $x_0\in B_{1/2}$ and a subspace V_0 such that $w(x_0,V_0)<-1+\theta$. Consider $v(x)=w(x,V_0)+1$, then $Lv\leq 0$ in B_1, $v\geq 0$ and $v(x_0)=\theta$. Note that $\inf_{B_{1/2}} v\leq\theta$.

Set $\Omega=\{v\leq\sqrt{\theta}\}=\{w(\cdot,V_0)\leq -1+\sqrt{\theta}\}$. Applying the L^ε estimate to $\hat{v}=v/\inf_{B_{1/2}} v$, we get

$$|\{v>\sqrt{\theta}\}\cap B_{1/4}|=|\{\hat{v}>\sqrt{\theta}/\inf_{B_{1/2}} v\}\cap B_{1/2}|\leq C\left(\frac{\sqrt{\theta}}{\inf_{B_{1/2}} v}\right)^{-\varepsilon}\leq C\theta^{\varepsilon/2}.$$

thus $|B_{1/4}\setminus\Omega|\leq C\theta^{\varepsilon/2}$.

In Ω, $1-\sqrt{\theta}\leq -w(x,V)\leq \mathrm{tr}D^2u(x)^-\leq 1$. On the other hand, denote $V^\perp$ be the orthogonal complement of V, then

$$w(x,V)+w(x,V^\perp)=\Delta u(x)=\mathrm{tr}D^2u(x)^+-\mathrm{tr}D^2u(x)^-.$$

Then,

$$0\leq \mathrm{tr}D^2u(x)^+-w(x,V^\perp)=w(x,V)+\mathrm{tr}D^2u(x)^-\leq\sqrt{\theta}.$$

Therefore, for θ small, we have

$$-w(x,V)\approx \mathrm{tr}D^2u(x)^-\approx\mathrm{tr}D^2u(x)^+\approx w(x,V^\perp)\approx 1,\qquad \text{in } \Omega.$$

Hence, there exists a universal $0<c<1$, such that $w(x,V^\perp)\geq c$ in Ω. Finally, we consider $h(x)=(c-w(x,V^\perp))^+$ in $B_{1/4}$. Then,

1) $Lh\geq 0$ in $B_{1/4}$;
2) $0\leq h\leq 2$ in $B_{1/4}$;
3) $v(0)=c$;
4) $v=0$ in Ω.

Applying the local maximum principle to h, we get

$$c\leq\sup_{B_{1/8}} h\leq C|\{h>0\}\cap B_{1/4}|\leq C|B_{1/4}\setminus\Omega|\leq C\theta^{\varepsilon/2}.$$

Taking θ sufficiently small, we get a contradiction. □

An easy consequence of the Evans-Krylov estimate is the following Liouville theorem.

Theorem 1.3. *Let F be a fully nonlinear, convex, uniformly elliptic operator with $F(0)=0$. Assume u is a smooth solution to $F(D^2u)=0$ in $\mathbb{R}^n$ with bounded Hessian. Then u is a quadratic polynomial.*

References

[CS10] Luis Caffarelli and Luis Silvestre, *On the Evans-Krylov theorem*, Proc. Amer. Math. Soc. **138** (2010), no. 1, 263–265. MR 2550191