9/11-25. **Maximum principle argument for Korevaar's estimate.**

Proposition 2. *Let $u \in C^3(B_1)$ be a smooth solution to the scalar curvature equation $\sigma_2(\kappa(u)) = f_0$ on B_1 for some constant $f_0 > 0$. Then we have the following gradient estimate:*

$$|Du(0)| \le C_1(n) \exp\left\{C_2(n)\left(1+\sqrt{f_0}\right)M^2\right\},$$

where $M = \sup_{B_1} u - u$.

Same as before, we compute using the moving orthonormal frame on the graph $\Sigma = (x, u(x))$. Let $\{E_1, \cdots, E_n, E_{n+1}\}$ be the standard orthonormal coordinates of $\mathbb{R}^{n+1}$, and $\{e_1, \cdots, e_n, \nu\}$ be an orthonormal frame on Σ. There are following well-known fundamental equations for hypersurfaces in $\mathbb{R}^{n+1}$:

- $X_i = e_i, \quad X_{ij} = -h_{ij}\nu$ (Gauss formula)
- $\nu_i = h_{ij}e_j$ (Weingarten equation)
- $h_{ijk} = h_{ikj}$ (Codazzi equation)
- $R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}$ (Gauss equation)

where h_{ij} is the second fundamental form of Σ, and $h_{ijk} = \nabla_{e_k}h_{ij}$, $h_{ijkl} = \nabla_{e_l}\nabla_{e_k}h_{ij}$. R_{ijkl} is the curvature tensor of Σ.

The function u can be viewed as a function on Σ by $u = \langle X, E_{n+1}\rangle$. Then

$$u_i = \langle e_i, E_{n+1}\rangle, \qquad u_{ij} = -h_{ij}\langle \nu, E_{n+1}\rangle = \frac{h_{ij}}{W}.$$

Under the orthonormal frame $\{e_1, \cdots, e_n, \nu\}$, the sigma-2 curvature equation is

$$\sigma_2(\kappa) = \sigma_2(h_{ij}) = \frac{1}{2}\left[H^2 - |A|^2\right] = \frac{1}{2}\left[\left(\sum_i h_{ii}\right)^2 - \sum_{i,j} h_{ij}^2\right] = f_0, \tag{1}$$

where $H = \sum_i h_{ii}$ is the mean curvature of Σ. Take the covariant derivative with respect to e_k, we get the linearized equation:

$$F^{ij}h_{ijk} = 0, \tag{2}$$

where $F^{ij} = \frac{\partial \sigma_2}{\partial h_{ij}} = H\delta_{ij} - h_{ij}$. Note that (F^{ij}) is positive definite, if $\lambda(h_{ij}) \in \Gamma_2$.

Now we are ready to prove Proposition 2.

Proof. By replacing u by $u - \sup_{B_1} u$, we may assume that $u \le 0$ and $u(0) = -u_0$ for some $u_0 \ge 0$.

Consider the following test function $P = \eta W$, where $W = \sqrt{1+|Du|^2} = -\frac{1}{\langle \nu, E_{n+1}\rangle}$ and $\eta = h \circ \varphi$ be a cutoff function, where

$$h(t) = e^{Kt} - 1, \text{for large } K \text{ to be fixed later}, \qquad \varphi = \left(\frac{u}{2u_0} + 1 - |x|^2\right)^+.$$

Since $|x|^2 = |X|^2 - u^2$, we can also view φ as a function on Σ. Hereafter, all computation are performed on Σ.

Claim (Jacobi inequality). We have $F^{ij}W_{ij} \ge 2F^{ij}W_iW_j/W$.

Proof. This result follows from straightforward computations. Compute the derivatives of W, we have

$$W_i = h_{ik} u_k W^2, \qquad \text{and} \qquad W_{ij} = h_{ijk} u_k W^2 + h_{ik} h_{jk} W + 2\frac{W_i W_j}{W}.$$

From the positive definiteness of (F^{ij}) and the linearized equation (2), we obtain

$$F^{ij} W_{ij} - 2\frac{F^{ij} W_i W_j}{W} \geq F^{ij} h_{ijk} u_k W^2 + F^{ij} h_{ik} h_{jk} W \geq 0.$$

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Suppose that P attains its positive maximum at $x_0 \in B_1$. By choosing $\{e_1, \cdots, e_n\}$ appropriately, we can assume that $\nabla u = u_1 e_1$ at x_0. Thus

$$u_i = \langle e_i, E_{n+1} \rangle = 0, \qquad \text{for } i = 2, \cdots, n,$$

and

$$u_1 = \langle e_1, E_{n+1} \rangle = \sqrt{1 - \langle \nu, E_{n+1} \rangle^2} = \sqrt{1 - \frac{1}{W^2}} = \frac{|Du|}{W}.$$

At the maximum point x_0, we have

$$0 = P_i = \eta_i W + \eta W_i, \tag{3}$$

and

$$0 \geq F^{ij} P_{ij} = (F^{ij} \eta_{ij}) W + 2F^{ij} \eta_i W_j + \eta F^{ij} W_{ij}. \tag{4}$$

Substituting (3) and the Jacobi inequality into (4), we obtain

$$0 \geq F^{ij} \eta_{ij}, \qquad \Longrightarrow \qquad 0 \geq K F^{ij} \varphi_i \varphi_j + F^{ij} \varphi_{ij}. \tag{5}$$

Next, we compute the derivatives of φ. Since x_0 is the positive maximum point, we know that φ is positive near x_0. Hence, $\varphi = u/2u_0 + 1 - |X|^2 + u^2$ near x_0. We have at x_0,

$$\begin{aligned} \varphi_i &= \frac{u_i}{2u_0} - 2\langle X, e_i \rangle + 2u u_i, \\ \varphi_{ij} &= \frac{u_{ij}}{2u_0} - 2\delta_{ij} + 2h_{ij} \langle X, \nu \rangle + 2u_i u_j + 2u u_{ij}. \end{aligned} \tag{6}$$

Therefore,

$$F^{ij} \varphi_{ij} = \underbrace{\frac{F^{ij} h_{ij}}{2u_0 W}}_{\geq 0} - 2\sum F^{ii} + 2F^{ij} h_{ij} \langle X, \nu \rangle + \underbrace{2F^{ij} u_i u_j}_{\geq 0} + 2\frac{u}{W} F^{ij} h_{ij}.$$

Since $\sum F^{ii} = (n-1)H$, $F^{ij} h_{ij} = 2f_0 > 0$ and

$$\langle X, \nu \rangle + \frac{u}{W} = \frac{x \cdot Du - u}{W} + \frac{u}{W} = \frac{x \cdot Du}{W} \geq -1.$$

We conclude that

$$F^{ij} \varphi_{ij} \geq -C(n)(H + f_0).$$

Claim. If $|Du| > A := \max\{8nu_0, 1\}$, then we have $F^{ij} \varphi_i \varphi_j \geq H/32u_0^2$.

Proof of Claim. Note that

$$F^{ij}\varphi_i\varphi_j = F^{11}\varphi_1^2 + 2\sum_{i\geq 2} F^{1i}\varphi_1\varphi_i + \underbrace{\sum_{i,j\geq 2} F^{ij}\varphi_i\varphi_j}_{\geq 0}.$$

For the first term. Since $X = \sum_{k=1}^{n}\langle X, E_k\rangle E_k + uE_{n+1}$, we know that

$$-\langle X, e_1\rangle + uu_1 = -\sum_{k=1}^{n}\langle X, E_k\rangle\langle E_k, e_1\rangle \underbrace{-u\langle e_1, E_{n+1}\rangle + uu_1}_{=0} = -\sum_{k=1}^{n}\langle X, E_k\rangle\langle E_k, e_1\rangle.$$

Moreover, for $k = 1, \cdots, n$, we have

$$\langle E_k, e_1\rangle^2 \leq 1 - \langle E_{n+1}, e_1\rangle^2 = 1 - \frac{|Du|^2}{W^2} = \frac{1}{W^2}.$$

Therefore, we conclude that

$$|-\langle X, e_1\rangle + uu_1| \leq \frac{1}{W}\sum_{k=1}^{n}|\langle X, E_k\rangle| \leq \frac{n}{W}.$$

From (6), we obtain

$$\varphi_1 = \frac{u_1}{2u_0} + 2(-\langle X, e_1\rangle + uu_1) \geq \frac{1}{W}\left(\frac{|Du|}{2u_0} - 2n\right) \geq \frac{1}{4\sqrt{2}u_0},$$

provided $Du \geq A := \max\{8nu_0, 1\}$. Then from the $DP = 0$ equation (3), we have

$$h_{11}u_1W^2 = W_1 = -\frac{\eta_1}{\eta}W = -\frac{Ke^{K\varphi}\varphi_1}{\eta}W < 0, \quad \Longrightarrow \quad h_{11} < 0.$$

Consequently, $F^{11} = H - h_{11} \geq H$, thus $F^{11}\varphi_1^2 \geq \frac{H}{32u_0^2}$.

For the second term, using the $DP = 0$ equation (3) again, for $i \geq 2$, we have

$$0 = Ke^{K\varphi}\varphi_i W + \eta h_{1i}u_1W^2, \qquad \Longrightarrow \qquad \varphi_i = -\frac{\eta u_1 W}{Ke^{K\varphi}}h_{1i}.$$

Therefore,

$$\sum_{i\geq 2} F^{1i}\varphi_1\varphi_i = \varphi_1\sum_{i\geq 2} -h_{1i}\varphi_i = \varphi_1\frac{\eta u_1 W}{Ke^{K\varphi}}\sum_{i\geq 2} h_{1i}^2 \geq 0.$$

The proof of claim is complete. #

Now (5) implies that

$$0 \geq K\frac{H}{32u_0^2} - C(n)(H + f_0).$$

If we take K sufficiently large, so that

$$K = C(n)(1 + \sqrt{f_0})u_0^2 > C(n)u_0^2\frac{H + f_0}{H},$$

this leads a contradiction. Thus the hypothesis in the above claim fails, it yields $|Du| \leq A$ at the maximum point x_0.

Finally, we conclude that

$$\eta(0)\sqrt{1+|Du(0)|^2} = P(0) \leq P(x_0) \leq (e^K - 1)\sqrt{1+A^2}.$$

Since $\eta_0 = e^{K/2} - 1$, we have

$$|Du(0)| \leq \underbrace{\frac{\sqrt{1+A^2}}{e^{K/2}-1}}_{\leq C(n)} e^K \leq C(n)\exp\{C(n)(1+\sqrt{f_0})u_0^2\}.$$

□