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Asymptotic Behavior of Exterior Minimal Graphs in Half Spaces

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Abstract

In this paper, we study the asymptotic behavior at infinity of solutions to the minimal surface equation in exterior domains of the half space. We prove that the solution u tends to a linear function with rate at least $x_n|x|^{-n}$. We further establish a higher order asymptotic expansion. Compared with existing results for exterior domains of the whole space, we do not require any growth condition or dimensional restriction.

Keywords Asymptotic behavior · Minimal surface equation · Half space · Blow-down

Mathematics Subject Classification 35J93 · 35B40 · 49Q05 · 53A10

1 Introduction

Over the last century, there has been tremendous interest in the Bernstein problem and the asymptotic behavior at infinity of minimal graphs. The Bernstein theorem states that any entire minimal graph over $\mathbb{R}^n$ must be a hyperplane. It was first proved for $n = 2$ by Bernstein [5], $n = 3$ by De Giorgi [12], $n = 4$ by Almgren [2] and $n \leq 7$ by Simons [35]. For $n \geq 8$, there are nonlinear entire minimal graphs constructed by Bombieri, De Giorgi and Giusti [4]. Under some additional hypotheses, there are many Liouville-type results. If the solution u has a bounded gradient, Moser [28] proved that u is linear by Harnack inequality; In [10, 13], u must be linear if we only require $|\nabla u(x)| = o(|x|)$.

For the minimal graphs over half spaces and general convex domains, Edelen and Wang [14] proved the following Bernstein-type result: any solution to the minimal surface equation over a convex domain with linear boundary condition must be linear.

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Unlike the entire case, their result does not need any growth hypothesis of u or any dimensional restriction, this is due to the linear boundary condition.

For general anisotropic minimal graphs or stable minimal hypersurfaces, see [27, 29, 36] [8, 9, 11, 15, 30] for the corresponding Bernstein results.

The asymptotic behavior of exterior minimal graphs was studied by many authors. Let u be a solution of the minimal surface equation outside a bounded subset in $\mathbb{R}^n$. For $n = 2$, Bers [6] proved that ∇u is bounded. Simon [40] extended this result for $n \leq 7$. Concerning the asymptotic expansion near infinity, Bers [6] proved that for $n = 2$, u tends to a linear function plus $c \log |x|$ at infinity with rate at least $|x|^{-1}$. For $n \geq 3$, under the additional hypothesis ∇u is bounded, using the asymptotic behavior of solutions of uniformly elliptic linear equations, see [16–18, 32–34, 41], u tends to a linear function with rate at least $|x|^{2-n}$. Furthermore, if $\nabla u \to 0$ as $|x| \to \infty$, Schoen [31] proved a more precise expansion, as $|x| \to \infty$,

$$u(x) = a + b|x|^{2-n} + \sum_{j=1}^{n} \frac{c_j x_j}{|x|^n} + O(|x|^{-n}),$$

for some constants $a, b, c_j \in \mathbb{R}$. Using Kelvin transforms, Han and Wang [21] improved this expansion to any order. Similar asymptotic expansions for solutions of other nonlinear elliptic equations, one can see, [7, 23] for the Monge-Ampère equation, [22] for the maximal surface equation, [26] for the special Lagrangian equation, and [25, 26] for general fully nonlinear equations.

The main purpose of this paper is to extend the above asymptotic expansion for solutions of the minimal surface equation to half spaces. We prove the following result:

Theorem 1.1 *Let $u \in C^2(\overline{\mathbb{R}}^n_+ \backslash B_1^+)$ be a solution of the minimal surface equation*

$$\begin{cases} \mathcal{M}u = \Delta u - \dfrac{u_i u_j}{1 + |\nabla u|^2} u_{ij} = 0 & \text{in } \mathbb{R}^n_+ \backslash \overline{B}_1^+, \\ u(x', 0) = l(x') = a + b' \cdot x' & \text{on } \partial\mathbb{R}^n_+ \backslash \partial B_1^+, \end{cases} \tag{1.1}$$

where $n \geq 2$, $\mathbb{R}^n_+ = \{(x', x_n) : x_n > 0\}$ and $l : \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \to \mathbb{R}$ is a linear function.

Then, as $|x| \to \infty$, we have

$$\begin{aligned} u(x) = a + b' \cdot x' + b_n x_n + \frac{c_n x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n} \\ + \sum_{j=1}^{n} \frac{d_j x_j x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^{n+2}} + O(|x|^{-n-\alpha}), \end{aligned} \tag{1.2}$$

for some constants $b_n, c_n, d_1, \cdots, d_n \in \mathbb{R}$ and any $\alpha \in (0, 1)$, where $b = (b', b_n)$.

Moreover, for any $m \geq 2$,

$$|\nabla u(x) - b| = O(|x|^{-n}), \quad |D^m u(x)| = O(|x|^{1-n-m}).$$

Here and hereafter, the notation $f = O(g)$ is used to indicate that $|f|$ is bounded by a constant multiple of $|g|$ for large $|x|$. We now make some remarks about our main theorem.

Remark 1.2 We emphasize that in our case of half spaces, we have no requirement on the dimension or the growth of solutions. This is due to the linear boundary condition and the rigidity of minimal graphs over half spaces.

Remark 1.3 One can not reduce our result to the asymptotic theorem in exterior domains of the whole space by reflection. Since the rate of solutions tending to a linear function at infinity in exterior domains of the half space is faster than in exterior domains of the whole space, this is due to the linear boundary condition. In particular, when $n = 2$, the logarithmic term vanishes.

Remark 1.4 We only focus on the asymptotic behavior of u at infinity, thus the half ball B_1^+ which we excluded in the theorem 1.1 is unimportant–it can be replaced by any compact subset of $\mathbb{R}^n_+$. Moreover, the regularity of u is also not critical. By the global regularity of the minimal surface equation and the standard bootstrap argument, the solution $u \in C^2(\overline{\mathbb{R}}^n_+ \backslash B_1^+)$ in fact implies that u is smooth up to the boundary in a smaller exterior domain, namely $u \in C^\infty(\overline{\mathbb{R}}^n_+ \backslash B_2^+)$.

Similar to Han-Wang [21], using the Kelvin-type transform, we can improve the asymptotic expansion (1.2) to higher order.

Theorem 1.5 *Let $u \in C^2(\overline{\mathbb{R}}^n_+ \backslash B_1^+)$ be a solution of (1.1). Then there exist a constant $b_n \in \mathbb{R}$ and a function w such that*

$$u(x) = a + b' \cdot x' + b_n x_n + |\sqrt{A^{-1}}x|^{2-n} w\left(\frac{\sqrt{A^{-1}}x}{|\sqrt{A^{-1}}x|^2}\right) \quad for\ x \in \overline{\mathbb{R}}^n_+ \backslash B_1^+. \tag{1.3}$$

and for some $r > 0$,

$$w = 0 \quad on\ \sqrt{A^{-1}}\partial\mathbb{R}^n_+ \cap B_r.$$

where $b = (b', b_n)$ and $A = I - \frac{b \otimes b}{1+|b|^2}$, $A^{-1} = I + b \otimes b$.

In addition, w is $C^{2,\alpha}$ near the origin for any $\alpha \in (0, 1)$. Moreover, for $n \geq 4$, if n is even, then $w \in C^\infty$; if n is odd, then $w \in C^{n-3,\alpha}$ for any $\alpha \in (0, 1)$.

Remark 1.6 From the Taylor expansion of w near the origin, we can improve the asymptotic expansion (1.2) to higher order.

Our idea of proof is inspired by [40] and [22] . We first prove the asymptotic expansion (1.2) under the additional linear growth condition, see Theorem 2.1. In this case, we only use tools from the theory of elliptic equations. We estimate the derivatives of u at infinity, then use the Bernstein-type result in half spaces to prove that any blow-down limit of u must be linear. Next, using the maximum principle and the Kelvin-type transforms, we prove the asymptotic expansion (1.2).

Without the linear growth condition, we now consider the blow-down of graph u in the sense of currents. Combining the compactness and Bernstein-type results for mass-minimizing currents, we can also prove that any blow-down limit of graph u must be a half plane. Then by the uniqueness theorem of tangent cones, see Allard-Almgren [1] or Simon [3, 37, 39], the limit half plane is unique; this half plane is so-called the tangent plane of graph u at infinity. Moreover, graph u can be expressed by the graph of some new function with sublinear growth over the tangent plane at infinity. By virtue of the asymptotic expansion for this new function, the asymptotic expansion (1.2) of the original solution u follows.

This paper is organized as follows. In Section 2, we derive (1.2) for solutions with linear growth and then prove Theorem 1.1 in Section 3.

2 Asymptotic Behavior for Solutions with Linear Growth

Throughout this paper, we adopt the following notations.

(i) $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$.
(ii) $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ is a ball in $\mathbb{R}^n$ and $D_r = \{y \in \mathbb{R}^{n+1} : |y| < r\}$ is a ball in $\mathbb{R}^{n+1}$.
(iii) $B_r^+(x) = B_r(x) \cap \mathbb{R}^n_+$. $B_r = B_r(0)$ and $B_r^+ = B_r^+(0)$.
(iii) $\Sigma_r(x) = B_r(x) \cap \partial\mathbb{R}^n_+$ and $\Sigma_r = B_r \cap \partial\mathbb{R}^n_+$.
(iv) $\mathcal{C}_r = \{(x', x_n) : |x'| < r,\ 0 < x_n < r\}$ be a cylinder, and $\Gamma_r = \partial\mathcal{C}_r \backslash \{x_n = 0\}$.
(v) For $i, j \in 1, 2, \cdots, n$, δ_{ij} is the Kronecker symbol, $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$.

Without loss of generality, up to subtracting some constant from u, we may assume that $a = 0$ and $l(x') = b' \cdot x'$.

The main result of this section is stated as follows.

Theorem 2.1 *Let $u \in C^2(\overline{\mathbb{R}}^n_+ \backslash B_1^+)$ be a solution of (1.1). Assume that u has at most linear growth, that is*

$$|u(x)| \leq K|x| \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B_1^+, \tag{2.1}$$

for some constant $K > 0$. Then the asymptotic expansion (1.2) and (1.3) hold.

Throughout this section, C denotes a positive constant depending only on n and K, whose meaning may be different from line to line. We divide the proof of Theorem 2.1 into three subsections.

2.1 Global C^1 estimates

In this subsection, we derive a global gradient bound in a smaller exterior domain. Our approach is modified from [24]. First, we use the Bernstein technique to reduce the global estimate to the boundary estimate.

Lemma 2.2 *Let $u \in C^2(\overline{\mathbb{R}}^n_+ \backslash B_1^+)$ be the solution of the minimal surface equation $\mathcal{M}u = 0$ in $\mathbb{R}^n_+ \backslash \overline{B}_1^+$. Assume that u satisfies the growth condition (2.1). Then, for any $\rho > 1$, we have*

$$\|\nabla u\|_{L^\infty(\overline{\mathbb{R}}^n_+ \backslash B_{4\rho}^+)} \leq C\left(1 + \|\nabla u\|_{L^\infty(\partial\mathbb{R}^n_+ \backslash \partial B_\rho^+)}\right), \tag{2.2}$$

where the universal constant $C > 0$ is independent on ρ.

Proof For any $x_0 \in \overline{\mathbb{R}}^n_+ \backslash B_{4\rho}^+$, set $r = \frac{|x_0|-1}{2}$. It is clear that $B_r^+(x_0) \subset \mathbb{R}^n_+ \backslash \overline{B}_\rho^+$. Depending on whether x_0 is closed to $\partial\mathbb{R}^n_+$, we split the proof into two cases.

Case I: Boundary case. If $\Sigma_r(x_0) := B_r(x_0) \cap \partial\mathbb{R}^n_+ \neq \varnothing$, then $\Sigma_r(x_0) \subset \partial\mathbb{R}^n_+ \backslash \partial B_\rho^+$. Set

$$v(x) = u(x) - \inf_{B_r^+(x_0)} u, \quad M := \sup_{B_r^+(x_0)} v$$

$$\gamma(t) = 1 + \frac{t}{M}, \quad \eta(x) = \left(1 - \frac{|x - x_0|^2}{r^2}\right)^2.$$

Consider the function

$$w(x) := \gamma(v(x))\eta(x) \log|\nabla v(x)|^2, \ x \in \Omega_r,$$

where $\Omega_r = \{x \in \overline{B}_r^+(x_0) : |\nabla u(x)| \geq 100\}$. Clearly, w is nonnegative in Ω_r and positive in the interior of Ω_r. Assume that w attains its positive maximum at $y_0 \in \Omega_r$. Since $\eta = 0$ on $\partial B_r(x_0)$, then y_0 cannot belong to $\partial\Omega_r \cap \partial B_r(x_0)$.

If $y_0 \in \partial\Omega_r$, then either $|\nabla u(y_0)| = |\nabla v(y_0)| = 100$ or $y_0 \in \Sigma_r(x_0)$. For the first case, we have

$$\log|\nabla u(x_0)|^2 \leq w(x_0) \leq w(y_0) \leq 2\log 100,$$

which implies $|\nabla u(x_0)| \leq C$. For the latter case, we have

$$\log|\nabla u(x_0)|^2 \leq w(x_0) \leq w(y_0) \leq 2\log\|\nabla u\|^2_{L^\infty(\Sigma_r(x_0))},$$

we deduce that $|\nabla u(x_0)| \leq C\|\nabla u\|_{L^\infty(\partial\mathbb{R}^n_+ \backslash \partial B_\rho^+)}$.

If $y_0 \in \mathrm{Int}(\Omega_r)$, from the calculation in [42] or [20, Theorem 3.3.1], we get

$$|\nabla u(x_0)| \leq \exp\left\{C\left(1 + \frac{M^2}{r^2}\right)\right\}.$$

Recalling that $M = \sup_{B_r^+(x_0)} u - \inf_{B_r^+(x_0)} u$ and $|x_0| = 2r + 1$. From the growth condition (2.1), we have $M \leq 2K(3r + 1)$, therefore, $|\nabla u(x_0)| \leq C$.

Case II: Interior case. If $B_r(x_0) \subset \mathbb{R}^n_+ \backslash \overline{B}_\rho^+$. Similar as above, using the interior gradient estimate of the minimal surface equation, see [42] or [20, Theorem 3.3.1], we can also get

$$|\nabla u(x_0)| \le \exp\left\{C\left(1+\frac{\omega^2}{r^2}\right)\right\},$$

where $\omega = \operatorname{osc}_{B_r^+(x_0)} u \le 2K(3r+1)$, hence $|\nabla u(x_0)| \le C$.

In summary, combining all cases, since $x_0 \in \overline{\mathbb{R}}^n_+ \backslash B^+_{4\rho}$ is arbitrary, we obtain (2.2). □

In the next lemma, we construct upper and lower barriers to bound the gradient of u on the boundary.

Lemma 2.3 *Let $u \in C^2(\overline{\mathbb{R}}^n_+ \backslash B_1^+)$ be a solution of (1.1) with linear growth (2.1). Then,*

$$\|\nabla u\|_{L^\infty(\partial\mathbb{R}^n_+ \backslash \partial B^+_{10})} \le C, \tag{2.3}$$

where $C > 0$ is a universal constant.

Proof By a rescaling argument, it suffices to prove that

$$\|\nabla u\|_{L^\infty(\partial\mathbb{R}^n_+ \cap \{10 \le |x| \le 20\})} \le C \tag{2.4}$$

In fact, if (2.4) holds, for any $\lambda < 1$, define $u_\lambda(x) = \lambda u(\lambda^{-1}x)$, then u_λ also satisfies (1.1) and (2.1). Applying (2.4), we get

$$\|\nabla u\|_{L^\infty(\partial\mathbb{R}^n_+ \cap \{10\lambda^{-1} \le |x| \le 20\lambda^{-1}\})} = \|\nabla u_\lambda\|_{L^\infty(\partial\mathbb{R}^n_+ \cap \{10 \le |x| \le 20\})} \le C$$

Notice that C is independent on λ, and $\lambda < 1$ is arbitrary, thus (2.3) holds.

Since u equals a linear function l on the flat boundary, then we only need to bound the normal derivative $\partial_n u$. Denote $\Sigma = \partial\mathbb{R}^n_+ \cap \{10 \le |x| \le 20\}$. For any $x_0 \in \Sigma$, same as [24], we construct a C^3 convex domain Ω around x_0. Precisely, let

$$\psi(t) = \frac{64}{35}\sqrt{t} - 2t^2 + \frac{8}{5}t^3 - \frac{3}{7}t^4, \quad \widetilde{\psi}(t) = \begin{cases} \psi(t), & t \in [0,1], \\ \psi(2-t), & t \in [1,2], \end{cases}$$

and let $\Omega = \{(x', x_n) : |x' - x_0| \le 2 + \widetilde{\psi}(x_n), 0 < x_n < 2\}$. Such Ω is a bounded C^3 convex domain around x_0.

Let $\eta \in C^\infty([0,\infty))$ satisfy $0 \le \eta \le 1$, $\eta \equiv 0$ in $[0,1]$ and $\eta \equiv 1$ in $[2,\infty)$. Define

$$\phi(x) = 100K\eta(|x-x_0|) + l(x)(1 - \eta(|x-x_0|)).$$

It is easy to check that $\phi \ge u$ on $\partial\Omega$. Since Ω is a C^3 convex domain and $\phi \in C^3$, refer to [20, Theorem 2.6.1] or [19, Theorem 16.10], the Dirichlet problem

$$\begin{cases} \mathcal{M}v = 0 & \text{in } \Omega, \\ v = \phi & \text{on } \partial\Omega \end{cases}$$

has a unique $C^2(\overline{\Omega})$ solution v. By the comparison principle, $u \leq v$ in $\overline{\Omega}$. Note that, $u = l = v$ on $\Sigma \cap \partial\Omega$, therefore

$$\partial_n u(x) \leq \partial_n v(x) \leq \|\nabla v\|_{L^\infty(\overline{\Omega})} \leq C, \quad \forall x \in \Sigma \cap \partial\Omega,$$

where $C > 0$ is universal. Similarly, we can construct a lower barrier to get the lower bound. Finally, we cover Σ by finitely many such domains $\Sigma \cap \partial\Omega$, hence we get $\|\partial_n u\|_{L^\infty(\Sigma)} \leq C$. □

Combining Lemma 2.2 and Lemma 2.3, we know that

$$\|\nabla u\|_{L^\infty(\overline{\mathbb{R}}^n_+ \backslash B^+_{40})} \leq C.$$

Therefore, the minimal surface equation becomes uniformly elliptic in an exterior domain.

2.2 Blow-down argument

It is well known that the blow-down limits reflect the behavior of u at infinity. In this subsection, we first prove a decay estimate of D^2u, then we use this estimate and the Bernstein-type result in half space to show that the blow-down limits of u must be linear.

Lemma 2.4 *Let $u \in C^2(\overline{\mathbb{R}}^n_+ \backslash B^+_1)$ be a solution of*

$$\begin{cases} \mathcal{M}u = 0 & \text{in } \mathbb{R}^n_+ \backslash \overline{B}^+_1, \\ u(x', 0) = l(x') & \text{on } \partial\mathbb{R}^n_+ \backslash \partial B^+_1, \\ |u(x)| \leq A|x|^\gamma & \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_1, \end{cases}$$

for some constant $A > 0$ and $\gamma \leq 1$. Then for $m = 1, 2$, we have

$$|D^m u(x)| \leq C|x|^{\gamma - m}, \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{100}, \tag{2.5}$$

where $C > 0$ depending only on n, A, γ.

Proof By the result in subsection 2.1, we have $\|\nabla u\|_{L^\infty(\overline{\mathbb{R}}^n_+ \backslash B^+_{40})} \leq C = C(n, A)$, thus the minimal surface equation is uniformly elliptic in $\overline{\mathbb{R}}^n_+ \backslash B^+_{40}$.

For any fixed $R > 100$, we will show that $|D^m u(x)| \leq CR^{\gamma - m}$ on $\partial B_R \cap \{x_n \geq 0\}$.

Case I: Near the boundary. For any $x_0 \in \partial B_R \cap \{x_n = 0\}$. Set

$$\eta(y) = \frac{4}{R} u\left(x_0 + \frac{R}{4} y\right), \quad y \in \overline{B}^+_2,$$

then $\eta \in C^2(\overline{B}_2^+)$ satisfies

$$\begin{cases} \mathcal{M}\eta = 0 & \text{in } B_2^+, \\ \eta(y', 0) = \widetilde{l}(y') & \text{on } \partial B_2^+ \cap \{x_n = 0\}, \end{cases}$$

where $\widetilde{l}$ equals l plus some constant. By the boundary $C^{1,\alpha}$ estimate for quasilinear equation, see [19, Chapter 13], there exist constants $\alpha \in (0, 1)$, $C > 0$ depending only on n, A, such that

$$\|\eta\|_{C^{1,\alpha}(\overline{B}_1^+)} \le C$$

Let $a^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$, then $\|a^{ij}(\nabla\eta)\|_{C^{0,\alpha}(\overline{B}_1^+)} \le C$ and $\lambda I \le a_{ij} \le \Lambda I$ in $\overline{B}_1^+$ for some universal constants $0 < \lambda \le \Lambda < \infty$. By Schauder estimate, for $m = 1, 2$, we have

$$4^{1-m} R^{m-1} \|D^m u\|_{L^\infty(B_{R/8}^+(x_0))} = \|D^m \eta\|_{L^\infty(B_{1/2}^+)} \le C\|\eta\|_{L^\infty(B_2^+)} \le C R^{\gamma-1}.$$

Hence, $|D^m u(x)| \le C R^{\gamma-m} \le C|x|^{\gamma-m}$ in $B_{R/8}^+(x_0)$.

Case II: Interior case. This case is easier. For any $x \in \partial B_R \cap \{x_n \ge R/8\}$. Set

$$\eta(y) = \frac{32}{R} u\left(x + \frac{R}{32} y\right), \quad y \in B_2.$$

Applying the interior $C^{1,\alpha}$ estimate, see [19, Chapter 13], and the Schauder estimate, we can also get

$$32^{1-m} R^{m-1} |D^m u(x)| = |D^m \eta(0)| \le C\|\eta\|_{L^\infty(B_2)} \le C R^{\gamma-1}.$$

Hence, $|D^m u(x)| \le C R^{\gamma-m} = C|x|^{\gamma-m}$ in $\partial B_R \cap \{x_n \ge R/8\}$.

Combining the above two cases, the proof is complete. □

Remark 2.5 By repeating the above argument to improve the regularity of η, we can get the corresponding decay estimates for the higher order derivatives $D^m u$, $m > 2$.

Now let $u \in C^2(\overline{\mathbb{R}}_+^n \backslash B_1^+)$ be a solution of (1.1) with linear growth (2.1). For any $\lambda > 0$, consider the rescaled function $u_\lambda(x) = \lambda u(\lambda^{-1}x)$, $x \in \overline{\mathbb{R}}_+^n \backslash B_\lambda^+$, then u_λ satisfies

$$\begin{cases} \mathcal{M}u_\lambda = 0 & \text{in } \mathbb{R}_+^n \backslash \overline{B}_\lambda^+, \\ u_\lambda(x') = l(x') = b' \cdot x' & \text{on } \partial\mathbb{R}_+^n \backslash \partial B_\lambda^+, \\ |D^m u_\lambda(x)| \le C|x|^{1-m} & \text{in } \overline{\mathbb{R}}_+^n \backslash B_{100\lambda}^+, \ m = 0, 1, 2. \end{cases} \tag{2.6}$$

Using the Arzela-Ascoli theorem and a standard diagonal argument, for any $\lambda_k \searrow 0$, there exists a subsequence (still denoted by $\{\lambda_k\}$), such that $\{u_{\lambda_k}\}$ converges to a function u_∞ in $C^1_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+\backslash\{0\})$. We claim that u_∞ is a linear function. In fact, from the $C^1_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+\backslash\{0\})$ convergence, $u_\infty \in C^1(\mathbb{R}^n_+)\cap C(\overline{\mathbb{R}}^n_+\backslash\{0\})$ solves the minimal surface equation weakly in $\mathbb{R}^n_+$. By the interior regularity of the minimal surface equation, $u_\infty \in C^2(\mathbb{R}^n_+)$. Note that each u_{λ_k} satisfies the linear growth condition $|u_{\lambda_k}(x)| \le C|x|$ in $\overline{\mathbb{R}}^n_+\backslash B^+_{100\lambda_k}$, passing to the limit, we get $|u_\infty(x)| \le C|x|$ in $\overline{\mathbb{R}}^n_+\backslash\{0\}$. Therefore, 0 is a removable singularity by defining $u_\infty(0) = 0$. Hence $u_\infty \in C^2(\mathbb{R}^n_+)\cap C(\overline{\mathbb{R}}^n_+)$ satisfies

$$\begin{cases}\mathcal{M}u_\infty = 0 & \text{in } \mathbb{R}^n_+,\\ u_\infty(x',0) = l(x') = b'\cdot x' & \text{on } \partial\mathbb{R}^n_+.\end{cases}$$

From the Liouville theorem in half space, see [14, 24], u_∞ is linear. Moreover, there exists $b_n \in \mathbb{R}$, such that $u_\infty(x) = b\cdot x = b'\cdot x' + b_n x_n$, where $b = (b', b_n)$. In the above argument, the blow-down limit obviously depends on the choice of $\{\lambda_k\}$. However, in this situation, the blow-down limit is unique.

Let $v(x) = u(x) - b\cdot x$, then v satisfies

$$\begin{cases}a^{ij}(\nabla u)\partial_{ij}v = 0 & \text{in } \mathbb{R}^n_+\backslash\overline{B}^+_{100},\\ v = 0 & \text{on } \partial\mathbb{R}^n_+\backslash\partial B^+_{100},\\ |v(x)| \le 2K|x| & \text{in } \overline{\mathbb{R}}^n_+\backslash B^+_{100},\end{cases} \tag{2.7}$$

where $a^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$. Since ∇u is bounded, we know that a_{ij} is uniformly elliptic in $\mathbb{R}^n_+\backslash\overline{B}^+_{100}$, with some universal ellipticity constants $0 < \lambda \le \Lambda < \infty$.

Lemma 2.6 *$\|v\|_{L^\infty(\overline{\mathbb{R}}^n_+\backslash B^+_{100})} \le C$ for some universal constant $C > 0$.*

Proof For any $r > 0$, let $\mathcal{C}_r = \{(x', x_n) : |x'| < r,\ 0 < x_n < r\}$ be a cylinder, and let $\Gamma_r := \partial\mathcal{C}_r\backslash\{x_n = 0\}$. Γ_r is the boundary of $\mathcal{C}_r$ except the bottom.

Since $u_{\lambda_k} \to b\cdot x$ in $C^1_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+\backslash\{0\})$, we have $\|\nabla v\|_{L^\infty(\Gamma_{\lambda_k^{-1}})} = \|\nabla u_{\lambda_k} - b\|_{L^\infty(\Gamma_1)} \to 0$. Therefore, for any $\varepsilon > 0$, there exists $\lambda_{k_\varepsilon} < 1/100$, such that

$$|\nabla v| < \varepsilon, \quad \text{on } \Gamma_{\lambda_{k_\varepsilon}^{-1}}.$$

On $\partial B^+_{100}\cap\{x_n > 0\}$, by the linear growth of v, we have $|v(x)| \le 200K$.

On $\Gamma_{\lambda_{k_\varepsilon}^{-1}}\cap\{|x'| = \lambda_{k_\varepsilon}^{-1}\}$, using $|\nabla v| < \varepsilon$ and $v|_{\{|x'|\ge 100, x_n=0\}} = 0$, we get $|v(x)| \le \varepsilon x_n$.

Similarly, on $\Gamma_{\lambda_{k_\varepsilon}^{-1}}\cap\{x_n = \lambda_{k_\varepsilon}^{-1}\}$, we have $|v(x)| \le \max_{\Gamma_{\lambda_{k_\varepsilon}^{-1}}\cap\{|x'|=\lambda_{k_\varepsilon}^{-1}\}}|v| + \varepsilon\lambda_{k_\varepsilon}^{-1} \le 2\varepsilon x_n$.

By the maximum principle, we obtain,

$$|v(x)| \le 200K + 2\varepsilon x_n \quad \text{in } \overline{\mathcal{C}}_{\lambda_{k_\varepsilon}^{-1}}\backslash B^+_{100}.$$

Sending ε to 0, we conclude $\|v\|_{L^\infty(\overline{\mathbb{R}}^n_+\backslash B^+_{100})} \le C$. ∎

Remark 2.7 By simply modifying the proof of Lemma 2.4, we deduce that

$$|\nabla u(x) - b| = |\nabla v(x)| \le C|x|^{-1} \text{ and } |D^2 v(x)| \le C|x|^{-2} \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{1000}. \tag{2.8}$$

In particular, ∇u tends to b at infinity, which implies the blow-down limit of u is unique and the tangent plane of graph u at infinity has the normal vector $\frac{(-b,1)}{\sqrt{1+|b|^2}}$.

2.3 Asymptotic expansion

Now we are ready to give the proof of Theorem 2.1. We first construct the upper barrier to prove the 0-order expansion, then improve it to higher order by a Kelvin-type transform.

Lemma 2.8 *Let $v(x) = u(x) - b \cdot x$ as above, then there exist universal constants $C > 0$, $R_1 \ge 1000$, such that*

$$|v(x)| = |u(x) - b \cdot x| \le \frac{Cx_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n} \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{R_1}.$$

Proof From (2.7), (2.8) and Lemma 2.6, v satisfies the following uniformly elliptic equation

$$\begin{cases} a^{ij}(\nabla u)\partial_{ij} v = 0 & \text{in } \mathbb{R}^n_+ \backslash \overline{B}^+_{1000}, \\ v = 0 & \text{on } \partial\mathbb{R}^n_+ \backslash \partial B^+_{1000}, \\ |v(x)| \le C & \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{1000}, \\ |\nabla v(x)| \le C|x|^{-1} & \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{1000}. \end{cases}$$

where $a^{ij}(\nabla u) = \delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}$. Since $|\nabla u(x) - b| = |\nabla v(x)| \le C|x|^{-1}$, the coefficient matrix tends to a constant matrix at infinity with the same rate, that is,

$$|a^{ij}(\nabla u(x)) - a^{ij}_\infty| \le C|x|^{-1} \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{1000}, \tag{2.9}$$

where $a^{ij}_\infty = \delta_{ij} - \frac{b_i b_j}{1+|b|^2}$. It is clear that $\lambda I \le (a^{ij}_\infty) \le \Lambda I$ for some universal constants $0 < \lambda \le \Lambda < \infty$. We now denote the uniformly elliptic operators $a^{ij}(\nabla u)\partial_{ij}$ and $a^{ij}_\infty \partial_{ij}$ by L and L_0 respectively.

We shall construct a barrier to bound v at infinity, our argument is modified from [18, 23]. Let

$$h(x) = \frac{x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n}.$$

From a direct computation, one can check that $L_0 h = 0$ in $\mathbb{R}^n_+ \backslash \{0\}$, and $h(x) \sim \frac{x_n}{|x|^n}$, $|\nabla h(x)| \sim |x|^{-n}$, $|D^2 h(x)| \sim |x|^{-n-1}$ in $\overline{\mathbb{R}}^n_+ \backslash B^+_{1000}$. Here the notation $f \sim g$ means $cg \leq f \leq Cg$ for some universal constant $c, C > 0$.

Consider $\overline{v} = h - h^\alpha$ for some $\alpha > 0$ to be fixed later, then

$$\begin{aligned}\partial_i \overline{v} &= \partial_i h - \alpha h^{\alpha-1} \partial_i h,\\ \partial_{ij} \overline{v} &= (1 - \alpha h^{\alpha-1}) \partial_{ij} h - \alpha(\alpha-1) h^{\alpha-2} \partial_i h \partial_j h.\end{aligned}$$

Thus $L_0 \overline{v} = (1-\alpha h^{\alpha-1}) L_0 h - \alpha(\alpha-1) h^{\alpha-2} a^{ij}_\infty \partial_i h \partial_j h$. We choose $\alpha \in (1, 2)$, then

$$\begin{aligned}L_0 \overline{v} &= -\alpha(\alpha-1) h^{\alpha-2} a^{ij}_\infty \partial_i h \partial_j h \leq -\alpha(\alpha-1)\lambda h^{\alpha-2} |\nabla h|^2\\ &\leq -C_1 \alpha(\alpha-1) \left(\frac{x_n}{|x|^n}\right)^{\alpha-2} \frac{1}{|x|^{2n}}\end{aligned}$$

Moreover,

$$\begin{aligned}|D^2 \overline{v}| &\leq C_2 \left(|D^2 h| + h^{\alpha-1} |D^2 h| + h^{\alpha-2} |Dh|^2\right)\\ &\leq C_2 \left(\frac{1}{|x|^{n+1}} + \left(\frac{x_n}{|x|^n}\right)^{\alpha-1} \frac{1}{|x|^{n+1}} + \left(\frac{x_n}{|x|^n}\right)^{\alpha-2} \frac{1}{|x|^{2n}}\right)\\ &\leq C_2 \left(\frac{1}{|x|^{n+1}} + \left(\frac{x_n}{|x|^n}\right)^{\alpha-2} \frac{1}{|x|^{2n}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}L\overline{v} &= (a^{ij}(\nabla u) - a^{ij}_\infty) \partial_{ij} \overline{v} + L_0 \overline{v}\\ &\leq C_2 \frac{1}{|x|} \left(\frac{1}{|x|^{n+1}} + \left(\frac{x_n}{|x|^n}\right)^{\alpha-2} \frac{1}{|x|^{2n}}\right) - C_1 \alpha(\alpha-1) \left(\frac{x_n}{|x|^n}\right)^{\alpha-2} \frac{1}{|x|^{2n}}\\ &= \frac{C_2}{|x|^{n+2}} + \left(\frac{C_2}{|x|} - C_1 \alpha(\alpha-1)\right) \frac{x_n^{\alpha-2}}{|x|^{n\alpha}}.\end{aligned}$$

If $|x| \geq \frac{2C_2}{C_1 \alpha(\alpha-1)}$, then

$$\begin{aligned}L\overline{v} &\leq \widetilde{C} \left(\frac{1}{|x|^{n+2}} - \frac{\alpha(\alpha-1)}{2} \frac{x_n^{\alpha-2}}{|x|^{n\alpha}}\right) \leq \frac{\widetilde{C}}{|x|^{n+2}} \left(1 - \frac{\alpha(\alpha-1)}{2} \frac{x_n^{\alpha-2}}{|x|^{n\alpha-n-2}}\right)\\ &\leq \frac{\widetilde{C}}{|x|^{n+2}} \left(1 - \frac{\alpha(\alpha-1)}{2} \frac{1}{|x|^{n\alpha-n-\alpha}}\right).\end{aligned}$$

Therefore, if $(n-1)\alpha - n < 0$, i.e. we fix some $\alpha \in \left(1, \frac{n}{n-1}\right)$, then there exists a universal constant $R_0 > 1000$, such that $L\overline{v} \leq 0$ in $\overline{\mathbb{R}}^n_+ \backslash B^+_{R_0}$.

For any $r > 0$, same as Lemma 2.6, let $\mathcal{C}_r$, Γ_r be the cylinder and its boundary without bottom respectively. Define $\mathcal{E}_r$ be the ellipsoid $\{|x|^2 + (b \cdot x)^2 < r^2\}$, and let $\mathcal{E}_r^+ = \mathcal{E}_r \cap \mathbb{R}^n_+$. We choose $R_1 > 1000$ satisfying $B^+_{R_0} \subset \mathcal{E}^+_{R_1}$. It is easy to see that $\mathcal{E}^+_{R_1} \subset B^+_{R_1}$.

On $\partial\mathbb{R}^n_+ \backslash \partial B^+_{R_0}$, we have $v = \overline{v} = 0$. Then on $\partial\mathcal{E}^+_{R_1} \cap \mathbb{R}^n_+$. By the boundedness of ∇v, we have $|v(x)| \le C x_n$. Now

$$\overline{v}(x) = \frac{x_n}{R_1^n} - \frac{x_n^\alpha}{R_1^{n\alpha}} \ge \widetilde{C}^{-1} x_n \ge \widetilde{C}^{-1}|v(x)| \quad \text{on } \partial\mathcal{E}^+_{R_1} \cap \mathbb{R}^n_+.$$

From the proof in Lemma 2.6, for any $\varepsilon > 0$, there exists a $R_\varepsilon > 1000$, such that $|\nabla v| \le \varepsilon$ on Γ_{R_ε}, thus $|v(x)| \le 2\varepsilon x_n$ on Γ_{R_ε}. Consequently,

$$\begin{cases} L(\widetilde{C}\overline{v} + 2\varepsilon x_n) \le Lv = 0 \le L(-\widetilde{C}\overline{v} - 2\varepsilon x_n) & \text{in } \mathcal{C}_{R_\varepsilon} \backslash \mathcal{E}^+_{R_1}, \\ -\widetilde{C}\overline{v} - 2\varepsilon x_n \le v \le \widetilde{C}\overline{v} + 2\varepsilon x_n & \text{on } \partial(\mathcal{C}_{R_\varepsilon} \backslash \mathcal{E}^+_{R_1}). \end{cases}$$

By the maximum principle, we get

$$|v(x)| \le \widetilde{C}\overline{v} + 2\varepsilon x_n \quad \text{in } \overline{\mathcal{C}}_{R_\varepsilon} \backslash \mathcal{E}^+_{R_1}.$$

Sending $\varepsilon \to 0$, we finally prove that

$$|v(x)| \le \widetilde{C}\overline{v} \le \widetilde{C} \frac{x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n} \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{R_1},$$

here we used the fact $\mathcal{E}^+_{R_1} \subset B^+_{R_1}$, the proof is complete. □

Remark 2.9 Same as Remark 2.7, applying the modified Lemma 2.4 with $\gamma = 1 - n$, we can improve (2.8) to

$$|\nabla u(x) - b| = |\nabla v(x)| \le C|x|^{-n} \text{ and } |D^2 v(x)| \le C|x|^{-n-1} \quad \text{in } \overline{\mathbb{R}}^n_+ \backslash B^+_{R_2}, \tag{2.10}$$

for some $R_2 > R_1$. Moreover, the asymptotic behavior of higher order derivatives follows from Remark 2.5.

Proof of Theorem 2.1 *Step 1. Kelvin type transform.* We first introduce some notations. Set

$$A(x) = (a^{ij}(\nabla u)) = I - \frac{\nabla u(x) \otimes \nabla u(x)}{1 + |\nabla u|^2} = \left(\delta_{ij} - \frac{u_i(x) u_j(x)}{1 + |\nabla u(x)|^2}\right),$$
$$A = (a^{ij}_\infty) = I - \frac{b \otimes b}{1 + |b|^2} = \left(\delta_{ij} - \frac{b_i b_j}{1 + |b|^2}\right).$$

One can easily check that $A^{-1} = I + b \otimes b = (\delta_{ij} + b_i b_j)$. From (2.10), we have $|A(x) - A| \le C|x|^{-n}$ in $\overline{\mathbb{R}}^n_+ \backslash B^+_{R_2}$. Let $H = \sqrt{A^{-1}}\mathbb{R}^n_+$ be a new half space, and $\partial H = \sqrt{A^{-1}}\partial\mathbb{R}^n_+$ be its boundary.

Introduce a new variable $z = \frac{\sqrt{A^{-1}}x}{|\sqrt{A^{-1}}x|^2}$, and let w be a function on $\overline{H} \cap B_{r_0}$ for some $r_0 > 0$ such that

$$u(x) = b \cdot x + |\sqrt{A^{-1}}x|^{2-n} w\left(\frac{\sqrt{A^{-1}}x}{|\sqrt{A^{-1}}x|^2}\right) \quad \text{for } x \in \overline{\mathbb{R}}^n_+ \backslash B^+_{R_2}. \tag{2.11}$$

Clearly, w is smooth except the origin. From a direct computation, we have

$$|z|^{n+2}\Delta w(z) = \operatorname{tr}(AD^2u(x))$$

Since $\operatorname{tr}(A(x)D^2u(x)) = 0$, we know that $f(z) := \Delta w(z)$ satisfies

$$|f(z)| \leq C|z|^{-n-2}|A(x) - A||D^2u(x)| \leq C|z|^{n-1} \quad \text{in } \overline{H} \cap \overline{B}_{r_1} \backslash \{0\},$$

for some $r_1 > 0$. Since $n \geq 2$, then $f \in C^{0,1}(\overline{H} \cap \overline{B}_{r_1})$. By the boundary Schauder estimates, we deduce that $w \in C^{2,\alpha}(\overline{H} \cap \overline{B}_{r_1/2})$ for any $\alpha \in (0, 1)$. Consequently,

$$\left|w(z) - w(0) - \nabla w(0) \cdot z - \frac{1}{2}z^T D^2 w(0) z\right| \leq C|z|^{2+\alpha} \quad \text{in } \overline{H} \cap \overline{B}_{\frac{r_1}{2}}. \tag{2.12}$$

Moreover, since $u(x) = b\cdot x$ on $\partial\mathbb{R}^n_+ \backslash \partial B^+_1$, we have $w = 0$ on $\partial H \cap B_\rho$, which implies that all tangential derivatives of w along ∂H at 0 must vanish. Denote $\varepsilon_i = \sqrt{A^{-1}}e_i$, $i = 1, 2, \cdots, n$, here $\varepsilon_1, \cdots, \varepsilon_{n-1}$ are tangential vectors of ∂H. Hence

$$w(0) = \nabla_{\varepsilon_i} w(0) = D^2_{\varepsilon_i \varepsilon_j} w(0) = 0, \quad i, j = 1, 2, \cdots, n-1.$$

Note that $|\sqrt{A^{-1}}x|^2 = x^T A^{-1} x = |x|^2 + (b \cdot x)^2$, then $z = \sum_{i=1}^{n} \frac{x_i \varepsilon_i}{|x|^2 + (b\cdot x)^2}$. Under the x coordinate, (2.12) becomes

$$\begin{aligned}\Bigg|\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^{n-2} (u(x) - b \cdot x) - \frac{\nabla_{\varepsilon_n} w(0) x_n}{|x|^2 + (b \cdot x)^2} + \sum_{j=1}^{n} \frac{D^2_{\varepsilon_j \varepsilon_n} w(0) x_j x_n}{\left(|x|^2 + (b \cdot x)\right)^2}\Bigg| \\ \leq C\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^{-2-\alpha}.\end{aligned}$$

Let $c_n = \nabla_{\varepsilon_n} w(0)$ and $d_j = D^2_{\varepsilon_j \varepsilon_n} w(0)$, $j = 1, 2 \cdots, n$, then we get

$$\left|u(x) - b \cdot x - \frac{c_n x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n} + \sum_{j=1}^{n} \frac{d_j x_j x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^{n+2}}\right| \leq C|x|^{-n-\alpha}$$

in $\overline{\mathbb{R}}^n_+ \backslash B^+_{R_3}$ for some $R_3 > R_2$ and any $\alpha \in (0, 1)$.

Step 2. Higher regularity of w. Using the relation (2.11) and the minimal surface equation for u, we compute the explicit equation that w satisfies.

Differentiating the relation (2.11), we have

$$u_i = b_i + |z|^{n-2} P_i, \qquad u_{ij} = |z|^{n-2} \left(Q_{ik} Q_{jl} w_{kl} + R_{ij} \right), \tag{2.13}$$

where

$$\begin{aligned} P_i &= |z|^2 \langle \varepsilon_i, \nabla w \rangle - [(n-2)w + 2\langle z, \nabla w \rangle] \langle \varepsilon_i, z \rangle, \qquad Q_{ik} = \varepsilon_{ik} |z|^2 - 2\langle \varepsilon_i, z \rangle z_k, \\ R_{ij} &= -[(n-2)w + w\langle z, \nabla w \rangle] |z|^2 \langle \varepsilon_i, \varepsilon_j \rangle + n[(n-2)w + 4\langle z, \nabla w \rangle] \langle \varepsilon_i, z \rangle \langle \varepsilon_j, z \rangle \\ &\quad - n|z|^2 [\langle \varepsilon_i, z \rangle \langle \varepsilon_j, \nabla w \rangle + \langle \varepsilon_j, z \rangle \langle \varepsilon_i, \nabla w \rangle]. \end{aligned}$$

Now the minimal surface equation $(1 + |\nabla u|^2)\Delta u - u_i u_j u_{ij} = 0$ becomes

$$\begin{aligned} 0 &= \left[\left(1 + \left| b + |z|^{n-2} P \right|^2 \right) \delta_{ij} - \left(b_i + |z|^{n-2} P_i \right) \left(b_i + |z|^{n-2} P_i \right) \right] u_{ij} \\ &= [(1 + |b|^2)\delta_{ij} - b_i b_j] u_{ij} \\ &\quad + |z|^{2n-4} \{ 2\langle b, P \rangle \delta_{ij} - b_i P_j - b_j P_i \} \{ Q_{ik} Q_{jl} w_{kl} + R_{ij} \} \\ &\quad + |z|^{3n-6} \{ |P|^2 \delta_{ij} - P_i P_j \} \{ Q_{ik} Q_{jl} w_{kl} + R_{ij} \}. \end{aligned}$$

Note that the first term $= (1 + |b|^2)\mathrm{tr}(AD^2 u) = (1 + |b|^2)|z|^{n+2}\Delta w$. After a lengthy computation and rearrangement, we finally get

$$\left\{ \begin{array}{l} (1 + |b|^2)\delta_{kl} \\ +|z|^{n-4} f_1^{kl}(z, w, \nabla w) \\ +|z|^{n-2} f_2^{kl}(z, w, \nabla w) \\ +|z|^{2n-4} f_3^{kl}(z, w, \nabla w) \end{array} \right\} w_{kl} = \left\{ \begin{array}{l} |z|^{n-4} f_4(z, w, \nabla w) \\ +|z|^{n-2} f_5(z, w, \nabla w) \\ +|z|^{2n-6} f_6(z, w, \nabla w) \\ +|z|^{2n-4} f_7(z, w, \nabla w) \end{array} \right\}. \tag{2.14}$$

Here f_i^{kl} and f_j are polynomials in terms of z, w and derivatives of w. Taking f_1^{kl} as an example, we have $f_1^{kl}(z, w, \nabla w) = -4(n-2)\sqrt{1 + |b|^2}\langle b, z \rangle z_k z_l w$. The expressions of other terms are too lengthy and therefore not explicitly presented here.

We notice that $|z|^{n-4}$ and $|z|^{n-2}$ are not smooth for odd n or small n. For $n \geq 4$, if n is even, then the coefficient and the right hand side of (2.14) are smoothly dependent on w. Recall that $w \in C^{2,\alpha}(\overline{H} \cap \overline{B}_{r_1/2})$ for any $\alpha \in (0, 1)$, By the boundary Schauder estimate and the bootstrap argument, u is smooth near the origin. If n is odd, since the term $|z|^{n-4}$ is $C^{n-5,1}$, then we can only improve the regularity of w to $C^{n-3,\alpha}(\overline{H} \cap \overline{B}_{r_2})$ for some $r_2 > 0$. $\square$

3 Proof of Theorem 1.1 and Theorem 1.5

Without the linear growth condition (2.1), we lack the decay estimates, namely Lemma 2.4, hence we cannot ensure the convergence of the blow-down sequence $\{u_\lambda\}$. Geometrically, this problem is not caused by a geometric defect in the surface we are studying, but rather by a defect in the function u that represents this surface. Therefore, we now focus on the surface itself, rather than its representation function u.

We first introduce some preliminaries about geometric measure theory, one can refer to [38] for details. Denote the k-dimensional Hausdorff measure by $\mathcal{H}^k$. Given an open subset U of $\mathbb{R}^{n+1}$, $\mathcal{D}^k(U)$ denotes the space of all smooth k-forms with compact support in U.

A k-current in U is a continuous linear functional on $\mathcal{D}^k(U)$, the set of all k-currents in U is denoted by $\mathcal{D}_k(U)$. The boundary of a k-current T is defined to be a $k-1$ current $\partial T \in \mathcal{D}_{k-1}(U)$ satisfying

$$\partial T(\omega) = T(d\omega), \quad \forall\, \omega \in \mathcal{D}^{k-1}(U).$$

A k-current T is said to be *integer multiplicity rectifiable* in U, if there is a countably k-rectifiable subset M_T of U, an $\mathcal{H}^k \llcorner M_T$-measurable simple unit k-vector ξ_T orienting $T_x M_T$ for $\mathcal{H}^k$-a.e. x, and an $\mathcal{H}^k \llcorner M_T$-measurable positive integer valued function θ_T, such that

$$T(\omega) = \int_{M_T} \langle \omega, \xi_T \rangle \theta_T \, \mathrm{d}\mathcal{H}^k, \quad \forall\, \omega \in \mathcal{D}^k(U).$$

Here the integer valued function θ_T is called the multiplicity function of T. In this case, the corresponding mass measure of T is $\mu_T = \theta_T \, \mathrm{d}\mathcal{H}^k \llcorner M_T$, i.e. for any open $W \subset U$,

$$\mathbb{M}_W(T) = \mu_T(W) = \int_{M_T \cap W} \theta_T \, \mathrm{d}\mathcal{H}^k.$$

An integer multiplicity rectifiable k-current T is said to be *integral* if both μ_T and $\mu_{\partial T}$ are Radon measures. An integral k-current T is said to be mass-minimizing in an open set $U' \subset U$, if for any $W \subset\subset U'$ and any integral k-current S in U' with $\partial S = 0$, $\mathrm{spt} S \subset W$, we have

$$\mathbb{M}_W(T) \le \mathbb{M}_W(T + S).$$

Remark 3.1 Let M be a k-dimensional oriented C^1 submanifold of an open set $U \subset \mathbb{R}^{n+1}$, and let the continuous function $\xi(x) = \pm\tau_1 \wedge \cdots \wedge \tau_k$ be the orientation of M, where $\tau_1, \cdots, \tau_k$ is an orthonormal basis for $T_x M$. Then there is a corresponding multiplicity 1 rectifiable k-current $[\![M]\!] \in \mathcal{D}_k(U)$ defined by

$$[\![M]\!](\omega) = \int_M \langle \omega, \xi \rangle \, \mathrm{d}\mathcal{H}^k, \quad \forall\, \omega \in \mathcal{D}^k(U).$$

Remark 3.2 Let E be a subdomain of $\mathbb{R}^{n+1}$, then $[\![E]\!] \in \mathcal{D}_{n+1}(U)$ is integral if and only if the characteristic function $\chi_E \in BV_{\mathrm{loc}}(U)$, or equivalently, E has locally finite perimeter in U.

Now we prove Theorem 1.1. Our idea is to view graph u as a multiplicity 1 integral current, then use the compactness theorem to extract a convergent subsequence of the blow-down of graph u. By the rigidity of mass-minimizing currents in half spaces, the limit must be a half hyperplane. Finally applying the uniqueness theorem of tangent cones and Theorem 2.1 to conclude the result.

Proof of Theorem 1.1 and Theorem 1.5 Let $G = \operatorname{graph} u = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n_+ \backslash \overline{B}^+_1\}$, Then G is a C^2 hypersurface in $\mathbb{R}^{n+1}$. Equipping G with upward orientation, then G can be viewed as a multiplicity 1 integral n-current in $\mathbb{R}^{n+1}$. Since u solves the minimal surface equation in $\mathbb{R}^n_+ \backslash \overline{B}^+_1$, then G is mass-minimizing in $(\mathbb{R}^n_+ \backslash \overline{B}^+_1) \times \mathbb{R}$. Moreover, G has the following form

$$G = \partial [\![U]\!] \llcorner \Big((\mathbb{R}^n_+ \backslash \overline{B}^+_1) \times \mathbb{R} \Big),$$

where $U = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n_+ \backslash \overline{B}^+_1, x_{n+1} < u(x)\}$.

By simply modifying the proof in [19, Chapter 16], we have the following upper volume bound: for any ball $D_r(y) \subset \mathbb{R}^{n+1}$, there holds $\mathcal{H}^n(G \cap D_r(y)) \leq Cr^n$, for some universal constant $C > 0$.

Similar as before, for any $\lambda > 0$, let $u_\lambda(x) = \lambda u(\lambda^{-1}x)$, and let $G_\lambda = \operatorname{graph} u_\lambda = \{(x, u_\lambda(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n_+ \backslash \overline{B}^+_\lambda\}$. Then G_λ can be also viewed as a multiplicity 1 integral n-current in $\mathbb{R}^{n+1}$ which is mass-minimizing in $(\mathbb{R}^n_+ \backslash \overline{B}^+_\lambda) \times \mathbb{R}$. Moreover, G_λ also has the form

$$G_\lambda = \partial [\![U_\lambda]\!] \llcorner \Big((\mathbb{R}^n_+ \backslash \overline{B}^+_\lambda) \times \mathbb{R} \Big),$$

where $U_\lambda = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n_+ \backslash \overline{B}^+_\lambda, x_{n+1} < u_\lambda(x)\}$. The upper volume bound also holds for G_λ.

Using the standard compactness theorem, see [38, Chapter 7], for any $\lambda_k \searrow 0$, there is a subsequence (still denoted by $\{\lambda_k\}$), and a multiplicity 1 integral n-current $T = \partial [\![V]\!] \llcorner \big(\mathbb{R}^n_+ \times \mathbb{R} \big)$ such that $T = \lim_k G_{\lambda_k}$ in the sensee of weak convergence of currents, and $\chi_{U_{\lambda_k}} \to \chi_V$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n_+ \times \mathbb{R})$, $\mathcal{H}^n \llcorner G_{\lambda_k} \to \mathcal{H}^n \llcorner \operatorname{spt} T$ in the sense of Radon measures. Moreover, T is mass-minimizing in $\mathbb{R}^n_+ \times \mathbb{R}$.

Note that u has linear boundary condition, i.e. $u(x', 0) = l(x')$ on $\partial \mathbb{R}^n_+ \backslash \partial B^+_1$, we extend l to the whole $\mathbb{R}^n$ by defining $l(x', x_n) = l(x')$. From [14, Lemma 2.2], each G_{λ_k} has the following form

$$G_{\lambda_k} = \big(\partial [\![E_{\lambda_k}]\!] - [\![L_{\lambda_k}]\!] \big) \llcorner \Big(\mathbb{R}^{n+1} \backslash (\overline{B}_{\lambda_k} \times \mathbb{R}) \Big),$$

where

$$E_{\lambda_k} = U_{\lambda_k} \cup \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} \backslash (\overline{B}_{\lambda_k} \times \mathbb{R}) : x_n < 0, x_{n+1} < l(x', x_n)\},$$
$$L_{\lambda_k} = \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} \backslash (\overline{B}_{\lambda_k} \times \mathbb{R}) : x_n < 0, x_{n+1} = l(x', x_n)\}.$$

Passing to the limit, we deduce that T has the form

$$T = \partial[\![\widetilde{E}]\!] - [\![\widetilde{L}]\!],$$

where

$$\begin{aligned}
&\widetilde{E} \cap (\mathbb{R}^n_+ \times \mathbb{R}) = V \cap (\mathbb{R}^n_+ \times \mathbb{R}),\\
&\widetilde{E} \backslash \left(\overline{\mathbb{R}}^n_+ \times \mathbb{R}\right) = \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_n < 0, x_{n+1} < l(x', x_n)\},\\
&\widetilde{L} = \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_n < 0, x_{n+1} = l(x', x_n)\}.
\end{aligned}$$

Therefore, by the Bernstein type theorem for mass-minimizing integral n-currents in half spaces with linear boundary, see [14, Theorem 1.2], we deduce that $T = [\![H]\!]$ for some half hyperplane H in $\mathbb{R}^{n+1}$. Applying the theorem of the uniqueness of tangent cones, see [1] or [3, 37, 39], the half hyperplane H is unique, i.e. it is independent on the choice of $\{\lambda_k\}$. Moreover, choosing a new coordinate on H, G can be expressed by the graph of a function h with sublinear growth over the exterior domain in H. To be specific, there exist sufficiently large $R_2 > R_1$, and a function $h \in C^2(H \backslash D_{R_1})$, such that

$$G \backslash D_{R_2} = \{y + h(y)\nu : y \in H \backslash D_{R_1}\} \backslash D_{R_2}, \tag{3.1}$$

where ν is the unit normal vector of H.

Claim 1 *The function h has sublinear growth, that is*

$$h(y) = o(|y|) \text{ or equivalently, } \lim_{|y| \to \infty} \frac{|h(y)|}{|y|} = 0.$$

Proof of Claim Following the argument in [37, Section 7] and [39, Part II, Section 5-6]. We express $G \backslash D_{R_2}$ by the "spherical image" of some function $\widetilde{h} \in C^2(H \backslash D_{R_1})$, that is,

$$G \backslash D_{R_2} = \left\{ \frac{y + \widetilde{h}(y)\nu}{\sqrt{1 + \frac{|\widetilde{h}(y)|^2}{|y|^2}}} : y \in H \backslash D_{R_1} \right\} \backslash D_{R_2}. \tag{3.2}$$

Set $\Sigma = H \cap \mathbb{S}^n$, under the polar coordinate $(\omega, \rho) = \left(\frac{y}{|y|}, |y|\right) \in \Sigma \times \mathbb{R}$ on H. Define $t = \log \frac{\rho}{R_1}$, and

$$\widetilde{u}(\omega, t) = \frac{\widetilde{h}(\rho\omega)}{\rho}, \quad \text{for } (\omega, t) \in \Sigma \times (1, \infty).$$

Now $\widetilde{u}$ solves a certain evolution equation that was investigated in [37, 39]. By [39, Theorem 5.7, P267-270], we have $u(\omega, t)$ tends to 0 as $t \to \infty$. This implies $\widetilde{h}$ has sublinear growth: $\widetilde{h}(\rho\omega) = o(\rho)$ as $\rho \to \infty$.

Next, we study the relation between expression (3.1) and the "spherical image" (3.2). Suppose $p \in G \backslash D_{R_2}$ can be expressed by

$$y + h(y)\nu = p = \frac{\widetilde{y} + \widetilde{h}(\widetilde{y})\nu}{\sqrt{1 + \frac{|\widetilde{h}(\widetilde{y})|^2}{|\widetilde{y}|^2}}}.$$

Comparing the coordinates, we get

$$y = \frac{\widetilde{y}}{\sqrt{1 + \frac{|\widetilde{h}(\widetilde{y})|^2}{|\widetilde{y}|^2}}}, \quad h(y) = \frac{\widetilde{h}(\widetilde{y})}{\sqrt{1 + \frac{|\widetilde{h}(\widetilde{y})|^2}{|\widetilde{y}|^2}}}.$$

From the first identity, we have $|y| \leq |\widetilde{y}|$. Hence

$$\frac{|h(y)|}{|y|} = \frac{|\widetilde{h}(\widetilde{y})|}{|\widetilde{y}|} = \widetilde{u}\left(\frac{\widetilde{y}}{|\widetilde{y}|}, |\widetilde{y}|\right) \to 0, \quad \text{as } |y| \to \infty.$$

Now the claim is proved.

Claim 2 *H can not be vertical, that is $\nu \cdot e_{n+1} \neq 0$.*

Proof of Claim Suppose to the contrary that H is a vertical half hyperplane, without loss of generality, we can assume $\nu = e_j$ for some $j \in \{1, 2, \cdots, n\}$. Since G is a minimal hypersurface in $(\mathbb{R}^n_+ \backslash \overline{B}^+_1) \times \mathbb{R}$, we know that h also satisfies the minimal surface equation under the coordinate on H. Since h has sublinear growth, by Theorem 2.1, we deduce that $h(y) = a + O(|y|^{1-n})$. In particular, h is bounded, which means that the height of G in the e_j direction must be bounded. It contradicts the definition of G. Since $G = \operatorname{graph} u$, and u is defined on $\overline{\mathbb{R}}^n_+ \backslash B^+_1$, then in the e_j direction, G must be extended to infinity. Therefore, the claim holds.

Since $\nu \cdot e_{n+1} \neq 0$, then we can assume that $H = \{(x', x_n, x_{n+1}) : x_n > 0,\ x_{n+1} = b \cdot x\}$, where $b = (b', b_n)$ for some $b_n \in \mathbb{R}$. Under the coordinate on H, h has the asymptotic expansion $h(y) = a + O(|y|^{1-n})$. In particular, h is bounded, which implies that u has at most linear growth. Applying Theorem 2.1, the proof is complete.

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Declarations

Conflict of interest. The authors declare they have no financial interests.

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