CONCAVITY OF THE SUPERCRITICAL SPECIAL LAGRANGIAN EQUATION

FAN ZHENYU

ABSTRACT. In this note, we show that the special Lagrangian equation $F(D^2u) = \sum_i \arctan\lambda_i(D^2u) = \Theta$ is concave when $\Theta \geq (n-2)\frac{\pi}{2}$. This result was first proved by Yuan [Yua06].

1. Introduction

Let u be a smooth solution to the special Lagrangian equation:

$$F(D^2u) = \sum_{i=1}^{n} \arctan\lambda_i(D^2u) = \Theta \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.1}$$

For any direction $e \in \mathbb{S}^{n-1}$, we differentiate (1.1) with respect to e, then we get the linearized equation:

$$F^{ij}u_{eij} = 0 \quad \text{in } \Omega, \text{ where } F^{ij} = \frac{\partial F}{\partial u_{ij}}(D^2u). \tag{1.2}$$

Denote $\Delta_F = F^{ij}\partial_{ij}$, it is called the linearized operator of F at u. Now (1.2) is equivalent to say that $\Delta_F u_e = 0$. Differentiating (1.2) with respect to e again, we get

$$F^{ij}u_{eeij} + F^{ij,kl}u_{eij}u_{ekl} = 0 \quad \text{in } \Omega, \text{ where } F^{ij,kl} = \frac{\partial^2 F}{\partial u_{ij}\partial u_{kl}}(D^2u). \tag{1.3}$$

The third order term $F^{ij,kl}u_{eij}u_{ekl}$ is the bad-term. However, if we know the sign of this term, then u_{ee} is the sub-/supersolution to the linearized operator Δ_F. This is crucial in the study of fully nonlinear elliptic equations.

The main result of this note is following:

Theorem 1.1 (Yuan)**.** *The third order term $A := F^{ij,kl}u_{eij}u_{ekl} \leq 0$, when $\Theta \geq (n-2)\frac{\pi}{2}$.*

Remark 1.2. By symmetry, $A \geq 0$ when $\Theta \leq (2-n)\frac{\pi}{2}$. The phase $|\Theta| = (n-2)\frac{\pi}{2}$ is called the critical phase.

Remark 1.3. Theorem 1.1 means that the Evans-Krylov estimate holds for (1.1) when $|\Theta| \geq (n-2)\frac{\pi}{2}$.

Date: April 15, 2025.

2. Derivatives of eigenvalues

Let u be a smooth function on Ω, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the ordered eigenvalues of D^2u. We shall compute the derivatives of λ_i with respect to the matrix item u_{ij} in this section.

Fix any $p \in \Omega$, by choosing a proper coordinate, we can assume that $D^2u(p)$ is diagonal, i.e. $D^2u(p) = \mathrm{diag}\{\lambda_1(p), \cdots, \lambda_n(p)\}$. We also assume that $\lambda_1(p) > \lambda_2(p)$. We only compute $\frac{\partial \lambda_1}{\partial u_{ij}}$ and $\frac{\partial^2 \lambda_1}{\partial u_{ij}\partial u_{kl}}$. The computations of derivatives of other eigenvalues are similar.

By the definition of eigenvalues, we have $0 = \det(D^2u - \lambda_1 I)$ in Ω. Let $\mathfrak{S}_n$ be the group of n-permutations, and let $D^2u - \lambda_1 I = (m_{ab})_{1\leq a,b\leq n}$, that is $m_{ab} = u_{ab} - \lambda_1\delta_{ab}$. Then

$$0 = \det(D^2u - \lambda_1 I) = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} m_{1\sigma(1)} \cdots m_{n\sigma(n)}. \tag{2.1}$$

- **1st order derivatives:** differentiating (2.1) with respect to u_{ij}, we get

$$0 = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} \sum_{a=1}^{n} \frac{\partial m_{a\sigma(a)}}{\partial u_{ij}} m_{1\sigma(1)} \cdots \widehat{m_{a\sigma(a)}} \cdots m_{n\sigma(n)}. \tag{2.2}$$

Here the sign $\widehat{\ }$ means that the term is omitted. Note that (m_{ab}) is diagonal and $m_{11} = 0$ at p. In order to ensure that the term $m_{1\sigma(1)} \cdots \widehat{m_{a\sigma(a)}} \cdots m_{n\sigma(n)}$ in the sum does not vanish, we must have $\sigma = Id$ and m_{11} is omitted. Therefore, at p, (2.2) becomes

$$0 = \frac{\partial m_{11}}{\partial u_{ij}} m_{22} \cdots m_{nn} = \frac{\partial(u_{11} - \lambda_1)}{\partial u_{ij}} (\lambda_2 - \lambda_1) \cdots (\lambda_n - \lambda_1).$$

Hence, we have at p,

$$\frac{\partial(u_{11} - \lambda_1)}{\partial u_{ij}} = 0, \quad \Longrightarrow \quad \frac{\partial \lambda_1}{\partial u_{ij}} = \frac{\partial u_{11}}{\partial u_{ij}} = \delta_{ij}^{11}. \tag{2.3}$$

- **2nd order derivatives:** differentiating (2.2) with respect to u_{kl} again, we get

$$\begin{aligned} 0 = &\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} \sum_{a=1}^{n} \frac{\partial^2 m_{a\sigma(a)}}{\partial u_{ij}\partial u_{kl}} m_{1\sigma(1)} \cdots \widehat{m_{a\sigma(a)}} \cdots m_{n\sigma(n)} \\ &+ \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} \sum_{a=1}^{n} \sum_{b\neq a} \frac{\partial m_{a\sigma(a)}}{\partial u_{ij}} \frac{\partial m_{b\sigma(b)}}{\partial u_{kl}} m_{1\sigma(1)} \cdots \widehat{m_{a\sigma(a)}} \cdots \widehat{m_{b\sigma(b)}} \cdots m_{n\sigma(n)}. \\ &= I + II. \end{aligned}$$

For I, similar as before, we have at p,

$$\begin{aligned} I = &\frac{\partial^2 m_{11}}{\partial u_{ij}\partial u_{kl}} m_{22} \cdots m_{nn} = \frac{\partial^2(u_{11} - \lambda_1)}{\partial u_{ij}\partial u_{kl}} (\lambda_2 - \lambda_1) \cdots (\lambda_n - \lambda_1) \\ = &- \frac{\partial^2 \lambda_1}{\partial u_{ij}\partial u_{kl}} (\lambda_2 - \lambda_1) \cdots (\lambda_n - \lambda_1). \end{aligned} \tag{2.4}$$

For II, in order to ensure that the term $m_{1\sigma(1)}\cdots\widehat{m_{a\sigma(a)}}\cdots\widehat{m_{b\sigma(b)}}\cdots m_{n\sigma(n)}$ in the sum does not vanish, we must have $\sigma(k)=k$ for $k\neq a,b$ and m_{11} is omitted. Then σ has only two choices: the identity Id and the swap $(a\,b)$. Besides, one of a,b must equal to 1. Therefore, at p,

$$II = 2\sum_{a>1}\frac{\partial m_{11}}{\partial u_{ij}}\frac{\partial m_{aa}}{\partial u_{kl}}m_{22}\cdots\widehat{m_{aa}}\cdots m_{nn} - \sum_{a>1}\frac{\partial m_{1a}}{\partial u_{ij}}\frac{\partial m_{a1}}{\partial u_{kl}}m_{22}\cdots\widehat{m_{aa}}\cdots m_{nn} - \sum_{a>1}\frac{\partial m_{a1}}{\partial u_{ij}}\frac{\partial m_{1a}}{\partial u_{kl}}m_{22}\cdots\widehat{m_{aa}}\cdots m_{nn}$$

By (2.3), the first term vanishes at p, then

(2.5)
$$II = -\sum_{a>1}\delta_{1i}\delta_{aj}\delta_{ak}\delta_{1l}\frac{(\lambda_2-\lambda_1)\cdots(\lambda_n-\lambda_1)}{\lambda_a-\lambda_1} - \sum_{a>1}\delta_{ai}\delta_{1j}\delta_{1k}\delta_{al}\frac{(\lambda_2-\lambda_1)\cdots(\lambda_n-\lambda_1)}{\lambda_a-\lambda_1}.$$

Combining (2.4) and (2.5), we have

$$0 = I + II = -\frac{\partial^2\lambda_1}{\partial u_{ij}\partial u_{kl}}(\lambda_2-\lambda_1)\cdots(\lambda_n-\lambda_1) - \sum_{a>1}\delta_{1i}\delta_{aj}\delta_{ak}\delta_{1l}\frac{(\lambda_2-\lambda_1)\cdots(\lambda_n-\lambda_1)}{\lambda_a-\lambda_1} - \sum_{a>1}\delta_{ai}\delta_{1j}\delta_{1k}\delta_{al}\frac{(\lambda_2-\lambda_1)\cdots(\lambda_n-\lambda_1)}{\lambda_a-\lambda_1}.$$

Hence,

$$\frac{\partial^2\lambda_1}{\partial u_{ij}\partial u_{kl}} = \sum_{a>1}\left(\delta_{1i}\delta_{aj}\delta_{ak}\delta_{1l}+\delta_{ai}\delta_{1j}\delta_{1k}\delta_{al}\right)\frac{1}{\lambda_1-\lambda_a} \tag{2.6}$$

$$= \begin{cases}\frac{1}{\lambda_1-\lambda_a}, & \text{if } i=l=1, j=k=a \text{ or } i=l=a, j=k=1;\\ 0, & \text{else.}\end{cases} \tag{2.7}$$

We summarize the above computations by the following proposition.

Proposition 2.1. *Let u be a smooth function on Ω, and let $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ be the ordered eigenvalues of D^2u. Assume that D^2u is diagonal at p, then*

$$\frac{\partial\lambda_a}{\partial u_{ij}} = \delta_{ij}^{aa} = \begin{cases}1 & i=j=a;\\ 0 & \textit{else.}\end{cases}$$

and

$$\frac{\partial^2\lambda_a}{\partial u_{ij}\partial u_{kl}} = \sum_{b\neq a}\left(\delta_{ai}\delta_{bj}\delta_{bk}\delta_{al}+\delta_{bi}\delta_{aj}\delta_{ak}\delta_{bl}\right)\frac{1}{\lambda_a-\lambda_b}$$
$$= \begin{cases}\frac{1}{\lambda_a-\lambda_b}, & \textit{if } i=l=a, j=k=b \textit{ or } i=l=b, j=k=a;\\ 0, & \textit{else.}\end{cases}$$

3. Proof of Theorem 1.1

We prove Theorem 1.1 pointwisely. Fixed any $p \in \Omega$, we may assume that $D^2u(p)$ is diagonal. We first compute $F^{ij}, F^{ij,kl}$ at p. By Proposition 2.1, at p, we have

$$F^{ij} = \sum_{a=1}^{n} \frac{\partial F}{\partial \lambda_a}\frac{\partial \lambda_a}{\partial u_{ij}} = \sum_{a=1}^{n} \frac{1}{1+\lambda_a^2}\delta_{ij}^{aa} = \begin{cases} \dfrac{1}{1+\lambda_i^2}, & i = j \\ 0, & else. \end{cases}$$

$$\begin{aligned} F^{ij,kl} &= \sum_{a,b=1}^{n} \frac{\partial^2 F}{\partial \lambda_a \partial \lambda_b}\frac{\partial \lambda_a}{\partial u_{ij}}\frac{\partial \lambda_b}{\partial u_{kl}} + \sum_{a=1}^{n} \frac{\partial F}{\partial \lambda_a}\frac{\partial^2 \lambda_a}{\partial u_{ij}\partial u_{kl}} \\ &= \begin{cases} \frac{-2\lambda_i}{(1+\lambda_i^2)^2}, & i=j=k=l; \\ \frac{-(\lambda_i+\lambda_j)}{(1+\lambda_i^2)(1+\lambda_j^2)} & i=k; j=l; and\ i \neq j \\ 0 & else \end{cases} \end{aligned}$$

Now the third order term $A = F^{ij,kl}u_{eij}u_{ekl}$ in Theorem 1.1 has the following form at p:

$$\begin{aligned} A = F^{ij,kl}u_{eij}u_{ekl} &= \sum_{i=1}^{n} F^{ii,ii}u_{eii}^2 + \sum_{i\neq j} F^{ij,ij}u_{eij}^2 \\ &= -2\sum_{i=1}^{n} \frac{\lambda_i}{(1+\lambda_i^2)^2}u_{eii}^2 - \sum_{i\neq j}\frac{\lambda_i+\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}u_{eij}^2 \\ &:= -2A_1 - A_2. \end{aligned}$$

Proof of Theorem 1.1. It suffices to show that $A_1, A_2 \geq 0$ when $\Theta \geq (n-2)\frac{\pi}{2}$, where $A_1 = \sum_i \frac{\lambda_i}{(1+\lambda_i^2)^2}u_{eii}^2$ and $A_2 = \sum_{i\neq j}\frac{\lambda_i+\lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}u_{eij}^2$.

If $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, the conclusion is obvious, then we assume $\lambda_n < 0$ in the following.

Claim 1. We have $\lambda_1 \geq \cdots \geq \lambda_{n-1} > 0 > \lambda_n$.

Let $\theta_i = \arctan \lambda_i$, by the equation (1.1), we have $\theta_1 + \cdots + \theta_n = \Theta \geq (n-2)\frac{\pi}{2}$. Note that $\theta_n = \arctan \lambda_n < 0$, if $\theta_{n-1} \leq 0$, then

$$\theta_1 + \cdots + \theta_n < \theta_1 + \cdots + \theta_{n-2} < (n-2)\frac{\pi}{2},$$

which leads a contradiction. #

Claim 2. For any $1 \leq i, j \leq n$ with $i \neq j$, we have $\lambda_i + \lambda_j \geq 0$.

To see this, it suffices to show that $\lambda_{n-1} + \lambda_n \geq 0$. Since $\theta_{n-1} + \theta_n = \Theta - (\theta_1 + \cdots + \theta_{n-2}) > \Theta - (n-2)\frac{\pi}{2} \geq 0$, and $\theta_{n-1} + \theta_n < \theta_{n-1} < \frac{\pi}{2}$, then

$$0 < \tan(\theta_{n-1} + \theta_n) = \frac{\lambda_{n-1} + \lambda_n}{1 - \lambda_{n-1}\lambda_n}.$$

Since $\lambda_{n-1}\lambda_n < 0$, then we get $\lambda_{n-1} + \lambda_n > 0$. #

From Claim 2, we can easily see that $A_2 \geq 0$. Next, we focus on A_1. Denote $t_i = u_{eii}$, then

$$A_1 = \sum_{i=1}^{n} \frac{\lambda_i}{(1+\lambda_i^2)^2} t_i^2 = \sum_{i=1}^{n} t_i^2 \tan\theta_i \cos^4\theta_i.$$

By the linearized equation (1.2), we have

$$0 = F^{ij} u_{eij} = \sum_{i=1}^{n} \frac{1}{1+\lambda_i^2} u_{eii} = \sum_{i=1}^{n} t_i \cos^2\theta_i.$$

Then, by the Cauchy inequality, we have

$$t_n^2 \cos^4\theta_n = \left(\sum_{i=1}^{n-1} t_i \cos^2\theta_i \right)^2 \leq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4\theta_i \tan\theta_i \right) \left(\sum_{i=1}^{n-1} \frac{1}{\tan\theta_i} \right).$$

Note that $\tan\theta_n < 0$, then

$$A_1 \geq \left(\sum_{i=1}^{n-1} t_i^2 \cos^4\theta_i \tan\theta_i \right) \left(1 + \sum_{i=1}^{n-1} \frac{\tan\theta_n}{\tan\theta_i} \right) = \left(\sum_{i=1}^{n-1} t_i^2 \cos^4\theta_i \tan\theta_i \right) \left(\sum_{i=1}^{n} \frac{\tan\theta_n}{\tan\theta_i} \right).$$

In order to prove $A_1 \geq 0$, we only need to show that $\sum_i \frac{1}{\tan\theta_i} \leq 0$. Let $\alpha_i = \frac{\pi}{2} - \theta_i$, then $\sum_i \frac{1}{\tan\theta_i} = \sum_i \tan\alpha_i$.

Claim 3. $\sum_i \tan\alpha_i \leq 0$.

Since we have assumed $\theta_n < 0$, then $(n-2)\frac{\pi}{2} \leq \Theta < \theta_1 + \cdots + \theta_{n-1} < (n-1)\frac{\pi}{2}$. Hence $\alpha_1 + \cdots + \alpha_n = n\frac{\pi}{2} - \Theta \in \left(\frac{\pi}{2}, \pi\right]$, thus we have

$$0 > \tan(\alpha_1 + \cdots + \alpha_n) = \frac{\tan(\alpha_1 + \cdots + \alpha_{n-1}) + \tan\alpha_n}{1 - \tan(\alpha_1 + \cdots + \alpha_{n-1})\tan\alpha_n}. \tag{3.1}$$

For $i = 1, \cdots, n-1$, we have $\theta_i \in \left(0, \frac{\pi}{2}\right)$, then $\alpha_i \in \left(0, \frac{\pi}{2}\right)$. Since $\theta_n \in \left(-\frac{\pi}{2}, 0\right)$, then $\alpha_n \in \left(\frac{\pi}{2}, \pi\right)$. Therefore,

$$0 < \alpha_1 + \cdots + \alpha_{n-1} = (\alpha_1 + \cdots + \alpha_n) - \alpha_n < \frac{\pi}{2},$$

which means $\tan(\alpha_1 + \cdots + \alpha_{n-1}) > 0$. Since $\tan\alpha_n < 0$, by (3.1), we have

$$\tan\alpha_n + \tan(\alpha_1 + \cdots + \alpha_{n-1}) < 0. \tag{3.2}$$

Note that

$$0 < \tan(\alpha_1 + \cdots + \alpha_{n-1}) = \frac{\tan(\alpha_1 + \cdots + \alpha_{n-2}) + \tan\alpha_{n-1}}{1 - \tan(\alpha_1 + \cdots + \alpha_{n-2})\tan\alpha_{n-1}}.$$

Since $\alpha_{n-1}, \alpha_1 + \cdots + \alpha_{n-2} \in \left(0, \frac{\pi}{2}\right)$, then we have $\tan(\alpha_1 + \cdots + \alpha_{n-2}), \tan\alpha_{n-1} > 0$. Hence $0 < 1 - \tan(\alpha_1 + \cdots + \alpha_{n-2})\tan\alpha_{n-1} < 1$, which means

$$\tan(\alpha_1 + \cdots + \alpha_{n-1}) > \tan(\alpha_1 + \cdots + \alpha_{n-2}) + \tan\alpha_{n-1}.$$

Repeating the above argument, we finally have

$$\begin{aligned} \tan(\alpha_1 + \cdots + \alpha_{n-1}) &> \tan(\alpha_1 + \cdots + \alpha_{n-2}) + \tan \alpha_{n-1} \\ &> \tan(\alpha_1 + \cdots + \alpha_{n-3}) + \tan \alpha_{n-2} + \tan \alpha_{n-1} \\ &> \cdots \\ &> \tan \alpha_1 + \cdots + \tan \alpha_{n-1}. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), the proof is complete. □

$\ln\left(\frac{1+x}{1-x}\right) + \ln\left(\frac{1+y}{1-y}\right) + \ln\left(\frac{1+z}{1-z}\right) = 100$

References

[Yua06] Yu Yuan, *Global solutions to special Lagrangian equations*, Proc. Amer. Math. Soc. **134** (2006), no. 5, 1355–1358. MR 2199179