BERNSTEIN THEOREM FOR SPECIAL LAGRANGIAN EQUATIONS IN PSEUDO EUCLIDEAN SPACE

FAN ZHENYU

1. Introduction

In pseudo Euclidean space $\mathbb{R}^{n,n} = \left(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{d}x^2 - \mathrm{d}y^2\right)$, the space like gradient graph $M = \{(x, Du(x)) : x \in \mathbb{R}^n\}$ is a maximal Lagrangian submanifold in $\mathbb{R}^{n,n}$ if u satisfies the following split special Lagrangian equation:

$$\sum_{i=1}^{n} \ln \frac{1+\lambda_i}{1-\lambda_i} = \text{const} = \kappa, \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

Theorem 1.1. *Let u be a space-like, smooth solution to (1.1), then u is a quadratic polynomial.*

Remark 1.2. We say u is space-like if $-I < D^2u < I$.

Proof. Introduce the new coordinate

$$\begin{cases} \overline{x} = \frac{1}{\sqrt{2}}(x-y), \\ \overline{y} = \frac{1}{\sqrt{2}}(x+y). \end{cases}$$

Since u is space-like, the gradient graph $M = \{(x, Du(x))\}$ is still a gradient graph of some new potential function $\overline{u}$ in $(\overline{x}, \overline{y})$ coordinate. Moreover,

$$D^2\overline{u} = (I + D^2u)(I - D^2u)^{-1} \quad \text{and} \quad \overline{\lambda}_i = \frac{1+\lambda_i}{1-\lambda_i}.$$

Now $\overline{u}$ is a convex solution to $\det D^2u = e^\kappa$ in $\mathbb{R}^n$. By the Bernstein theorem for Monge-Ampère equations due to Jörgens-Calabi-Pogerelov, we have $\overline{u}$ is a quadratic polynomial. Hence, M is flat. This implies that the original u is also a quadratic polynomial. □

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