10 多元函数的微分-2

知识点回顾:

- 复合函数求导;
- 方向导数与梯度, 常用的微分算子;
- 多元函数的中值定理和 Taylor 展开.

问题 10.1. *设 f 是 $\mathbb{R}^3$ 上的函数, 且设 e_1, e_2, e_3 是 $\mathbb{R}^3$ 中任意 3 个互相垂直的方向. 证明:*

$$\left(\frac{\partial f}{\partial e_1}\right)^2 + \left(\frac{\partial f}{\partial e_2}\right)^2 + \left(\frac{\partial f}{\partial e_3}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2,$$
$$\frac{\partial^2 f}{\partial e_1^2} + \frac{\partial^2 f}{\partial e_2^2} + \frac{\partial^2 f}{\partial e_3^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

RMK. 这个题目说明的是 $|\nabla f|^2$ 和 Δf 不依赖于坐标系的选取.

Proof. 设 $e_1 = (a_{11}, a_{12}, a_{13})^T, e_2 = (a_{21}, a_{22}, a_{23})^T, e_3 = (a_{31}, a_{32}, a_{33})^T$. 因 e_1, e_2, e_3 互相垂直, 故矩阵

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

是一个正交矩阵, 即 $AA^T = A^T A = I$. 或者用求和的语言来说, 有 $\sum_{i=1}^3 a_{ki} a_{li} = \delta_{kl} = \begin{cases} 1, \text{ 如果} k = l; \\ 0, \text{ 如果} k \neq l. \end{cases}$

此时, 记 $x_1 = x, x_2 = y, x_3 = z$. 有

$$\begin{aligned} \sum_{i=1}^3 \left(\frac{\partial f}{\partial e_i}\right)^2 &= \sum_{i=1}^3 \left(\sum_{k=1}^3 a_{ik} \frac{\partial f}{\partial x_k}\right)^2 = \sum_{i=1}^3 \left(\sum_{k=1}^3 a_{ik} \frac{\partial f}{\partial x_k}\right) \left(\sum_{l=1}^3 a_{il} \frac{\partial f}{\partial x_l}\right) \\ &= \sum_{i,k,l=1}^3 a_{ik} a_{il} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_l} = \sum_{k,l=1}^3 \delta_{kl} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_l} \\ &= \sum_{k=1}^3 \left(\frac{\partial f}{\partial x_k}\right)^2. \end{aligned}$$

对于二阶导数, 有

$$\frac{\partial^2 f}{\partial e_i^2} = \sum_{k,l=1}^3 a_{ik} a_{il} \frac{\partial^2 f}{\partial x_k \partial x_l}.$$

同样,

$$\sum_{i=1}^3 \frac{\partial^2 f}{\partial e_i^2} = \sum_{i,k,l=1}^3 a_{ik} a_{il} \frac{\partial^2 f}{\partial x_k \partial x_l} = \sum_{k,l=1}^3 \delta_{kl} \frac{\partial^2 f}{\partial x_k \partial x_l} = \sum_{k=1}^3 \frac{\partial^2 f}{\partial x_k^2}.$$

□

问题 10.2. 设 f 是 $\mathbb{R}^2$ 上的函数, 证明：在极坐标下梯度有如下表示

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{r}_0 + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta}_0,$$

其中 $\boldsymbol{r}_0, \boldsymbol{\theta}_0$ 分别是径向和圆周方向的单位向量.

Proof. 由向量分解, 有

$$\nabla f = (\nabla f \cdot \mathbf{r}_0)\mathbf{r}_0 + (\nabla f \cdot \boldsymbol{\theta}_0)\boldsymbol{\theta}_0.$$

因 $\mathbf{r}_0 = (\cos\theta, \sin\theta), \boldsymbol{\theta}_0 = \left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right) = (-\sin\theta, \cos\theta)$, 故

$$\begin{aligned}
\nabla f \cdot \mathbf{r}_0 &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = \frac{\partial f}{\partial r}. \\
\nabla f \cdot \boldsymbol{\theta}_0 &= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \\
&= \frac{1}{r}\left[\frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}r\cos\theta\right] = \frac{1}{r}\frac{\partial f}{\partial \theta}.
\end{aligned}$$

因此,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r}_0 + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta}_0.$$

□

问题 10.3. (1) 求函数 $f(x, y) = \arctan\frac{x}{y}$ 在 (a, b) 点附近的二次 *Taylor* 展开, 其中 $a, b > 0$.

(2) 求函数 $f(x, y) = xe^{x+y}$ 在原点附近的三次 *Taylor* 展开.

Proof. (1) 我们直接计算 f 在 (a, b) 处的各阶偏导数. 先计算一阶偏导数:

$$\frac{\partial f}{\partial x}(x, y) = \frac{y}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = -\frac{x}{x^2 + y^2}.$$

在 (a, b) 点处, 有

$$\nabla f(a, b) = \left(\frac{b}{a^2 + b^2}, -\frac{a}{a^2 + b^2}\right).$$

再求 (a, b) 点处的二阶导数, 有

$$D^2 f(a, b) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial x^2} \end{pmatrix} = \begin{pmatrix} -\dfrac{2ab}{(a^2 + b^2)^2} & \dfrac{a^2 - b^2}{(a^2 + b^2)^2} \\ \dfrac{a^2 - b^2}{(a^2 + b^2)^2} & \dfrac{2ab}{(a^2 + b^2)^2} \end{pmatrix}.$$

因此

$$\begin{aligned}
f(x, y) &= f(a, b) + \nabla f(a, b) \cdot (x - a, y - b) + \frac{1}{2}(x - a, y - b) D^2 f(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix} \\
&= \arctan\frac{a}{b} + \frac{b}{a^2 + b^2}(x - a) - \frac{a}{a^2 + b^2}(y - b) - \frac{ab}{(a^2 + b^2)^2}(x - a)^2 \\
&\quad + \frac{a^2 - b^2}{(a^2 + b^2)^2}(x - a)(y - b) + \frac{ab}{(a^2 + b^2)^2}(y - b)^2 + o(x^2 + y^2).
\end{aligned}$$

(2) 利用一元函数 e^t 在原点处的 Taylor 展开, 有

$$e^{x+y} = 1 + (x+y) + \frac{(x+y)^2}{2} + \frac{(x+y)^3}{6} + o\left((x^2+y^2)^{\frac{3}{2}}\right).$$

两边同乘 x, 得到

$$\begin{aligned} xe^{x+y} &= x + x(x+y) + \frac{x(x+y)^2}{2} + \frac{x(x+y)^3}{6} + o\left(x(x^2+y^2)^{\frac{3}{2}}\right) \\ &= x + x^2 + xy + \frac{x^3}{2} + x^2y + \frac{xy^2}{2} + o\left((x^2+y^2)^{\frac{3}{2}}\right). \end{aligned}$$

□

问题 10.4. *设二元函数 $f(u,v), g(u,v)$ 满足 Cauchy-Riemann 方程:*

$$\begin{cases} \frac{\partial f}{\partial u} = \frac{\partial g}{\partial v} \\ \frac{\partial f}{\partial v} = -\frac{\partial g}{\partial u} \end{cases}.$$

又设二元函数 w 满足 Laplace 方程 $\Delta w = 0$.

(1) 考虑 $\widetilde{w}(u,v) = w(f(u,v), g(u,v))$, 证明 $\widetilde{w}$ 也满足 Laplace 方程, 即 $\dfrac{\partial^2 \widetilde{w}}{\partial u^2} + \dfrac{\partial^2 \widetilde{w}}{\partial v^2} = 0$.

(2) 证明: fg 也满足 Laplace 方程, 即 $\dfrac{\partial^2}{\partial u^2}(fg) + \dfrac{\partial^2}{\partial v^2}(fg) = 0$.

Proof. (1) 由链式法则, 有

$$\begin{aligned} \widetilde{w}_u &= w_1 f_u + w_2 g_u, \\ \widetilde{w}_v &= w_1 f_v + w_2 g_v. \end{aligned}$$

再求二阶导数, 得

$$\begin{aligned} \widetilde{w}_{uu} &= (w_{11} f_u + w_{12} g_u) f_u + w_1 f_{uu} + (w_{21} f_u + w_{22} g_u) g_u + w_2 g_{uu}, \\ \widetilde{w}_{vv} &= (w_{11} f_v + w_{12} g_v) f_v + w_1 f_{vv} + (w_{21} f_v + w_{22} g_v) g_v + w_2 g_{vv}. \end{aligned}$$

因此

$$\begin{aligned} \Delta\widetilde{w} &= \widetilde{w}_{uu} + \widetilde{w}_{vv} \\ &= w_{11}(f_u^2 + f_v^2) + 2w_{12}(f_u g_u + f_v g_v) + w_{22}(g_u^2 + g_v^2) + w_1(f_{uu} + f_{vv}) + w_2(g_{uu} + g_{vv}) \end{aligned}$$

由 Cauchy-Riemann 方程可知, $f_{uu} + f_{vv} = g_{uu} + g_{vv} = 0$, $f_u g_u + f_v g_v = 0$, 而且 $f_u^2 + f_v^2 = g_u^2 + g_v^2$. 因此

$$\Delta\widetilde{w} = (w_{11} + w_{22})(f_u^2 + f_v^2) = 0.$$

(2) 用 Lebniz 公式, 有

$$\begin{aligned} (fg)_u &= f_u g + f g_u, \\ (fg)_v &= f_v g + f g_v. \end{aligned}$$

再求二阶导数, 得

$$(fg)_{uu} = f_{uu}g + 2f_u g_u + f g_{uu},$$
$$(fg)_{vv} = f_{vv}g + 2f_v g_v + f g_{vv}.$$

因此,

$$\Delta(fg) = (f_{uu} + f_{vv})g + 2(f_u g_u + f_v g_v) + f(g_{uu} + g_{vv}) = 0.$$

□

问题 10.5. 设 φ, ψ 是任意两个光滑一元函数, 对于 $(x,t) \in \mathbb{R}^2$, 定义函数 $u(x,t) = \varphi(x-at)+\psi(x+at)$. 证明 $u(x,t)$ 满足波动方程:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Proof. 直接求导, 得

$$u_t = \varphi'(x-at)\cdot(-a) + \psi'(x+at)\cdot a, \quad u_{tt} = \varphi''(x-at)\cdot a^2 + \psi''(x+at)\cdot a^2.$$
$$u_x = \varphi'(x-at) + \psi'(x+at), \quad u_{xx} = \varphi''(x-at) + \psi''(x+at).$$

结论显然. □

问题 10.6 (Bochner 公式)**.** 证明:

$$\frac{1}{2}\Delta|\nabla u|^2 = |D^2 u|^2 + \nabla u \cdot \nabla(\Delta u).$$

特别的, 如果 u 是调和函数, 这时我们能得到 $\Delta|\nabla u|^2 \geq 0$.

Proof. 为简单起见, 我们只证二元函数的情况. 记 $v = |\nabla u|^2 = u_1^2 + u_2^2$, 则

$$v_1 = 2(u_1 u_{11} + u_2 u_{12}),$$
$$v_2 = 2(u_1 u_{12} + u_2 u_{22}).$$

再求二阶导数,

$$v_{11} = 2(u_{11}^2 + u_1 u_{111} + u_{12}^2 + u_2 u_{112}),$$
$$v_{22} = 2(u_{12}^2 + u_1 u_{122} + u_{22}^2 + u_2 u_{222}).$$

因此,

$$\begin{aligned}\frac{1}{2}\Delta v &= \frac{1}{2}(v_{11} + v_{22}) \\ &= (u_{11}^2 + 2u_{12}^2 + u_{22}^2) + u_1(u_{111} + u_{221}) + u_2(u_{112} + u_{222}) \\ &= |D^2 u|^2 + (u_1, u_2)\cdot(v_1, v_2) = |D^2 u|^2 + \nabla u \cdot \nabla v.\end{aligned}$$

□

问题 10.7 (Liouville 定理)**.** 设 $u(x,y)$ 是全空间 $\mathbb{R}^2$ 上的调和函数, 而且 u 有界. 证明 u 一定为常函数.