5 不定积分和定积分的计算

知识点回顾:

- 有理函数的不定积分: 标准做法. 以及其他一类可以通过换元可以变成有理函数的三角不定积分.
- 定积分的计算: 换元、分部积分. * 瑕积分和反常积分.
- 定积分的应用.

问题 5.1. 计算下列不定积分:

(1) $\int \frac{x^5+x^4-8}{x^3-4x}\,\mathrm{d}x$;

(2) $\int \frac{1}{8-4\sin x+7\cos x}\,\mathrm{d}x$;

(3) $\int \frac{1}{\sin 2x+2\sin x}\,\mathrm{d}x$;

(4) $\int \frac{x+1}{x\sqrt{x-2}}\,\mathrm{d}x$.

Solutions. (1) *Step 1.* 先用多项式除法把假分式化成多项式加真分式.

$$x^5+x^4-8=(x^2+x+4)(x^3-4x)+4x^2+16x-8 \quad\Longrightarrow\quad \frac{x^5+x^4-8}{x^3-4x}=x^2+x+4+\frac{4(x^2+4x-2)}{x(x+2)(x-2)}.$$

Step 2. 利用待定系数法分解真分式. 设 $\frac{4(x^2+4x-2)}{x(x+2)(x-2)}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}$. 通分, 得

$$A(x+2)(x-2)+Bx(x-2)+Cx(x+2)=4x^2+16x-8.$$

比较两端系数, 得

$$\begin{cases} A+B+C=4 \\ 2C-2B=16 \\ -4A=-8 \end{cases} \qquad\Longrightarrow\qquad \begin{cases} A=2 \\ B=-3 \\ C=5 \end{cases}.$$

Step 3. 计算结果. 因此,

$$\begin{aligned}\int \frac{x^5+x^4-8}{x^3-4x}\,\mathrm{d}x &= \int x^2+x+4+\frac{2}{x}+\frac{-3}{x+2}+\frac{5}{x-2}\,\mathrm{d}x \\ &= \frac{1}{3}x^3+\frac{1}{2}x^2+4x+2\log|x|-3\log|x+2|+5\log|x-2|+C.\end{aligned}$$

(2) 做万能替换: $t=\tan\frac{x}{2}$, 则 $\sin x=\frac{2t}{1+t^2}, \cos t=\frac{1-t^2}{1+t^2}, \mathrm{d}x=\frac{2}{1+t^2}\,\mathrm{d}t$. 因此,

$$\begin{aligned}\int \frac{1}{8-4\sin x+7\cos x}\,\mathrm{d}x &= \int \frac{1}{8-\frac{8t}{1+t^2}+\frac{7(1-t^2)}{1+t^2}}\cdot\frac{2\,\mathrm{d}t}{1+t^2}=2\int\frac{1}{15-8t+t^2}\,\mathrm{d}t \\ &= 2\int\frac{1}{(t-3)(t-5)}\,\mathrm{d}t=\int\frac{1}{t-5}-\frac{1}{t-3}\,\mathrm{d}t \\ &= \log|t-5|-\log|t-3|+C=\log\left|\tan\frac{x}{2}-5\right|+\log\left|\tan\frac{x}{2}-3\right|+C.\end{aligned}$$

(3) 注意到,

$$
\begin{aligned}
\int \frac{1}{\sin 2x + 2\sin x}\,\mathrm{d}x &= \int \frac{1}{2\sin x(1+\cos x)}\,\mathrm{d}x = -\frac{1}{2}\int \frac{\mathrm{d}(\cos x)}{\sin^2 x(1+\cos x)} \\
(\text{做换元 } t=\cos x) \quad &= -\frac{1}{2}\int \frac{1}{(1-t^2)(1+t)}\,\mathrm{d}t \\
&= \frac{1}{8}\log\left|\frac{1-t}{1+t}\right| + \frac{1}{4(1+t)} + C \\
&= \frac{1}{8}\log\left|\frac{1-\cos x}{1++\cos x}\right| + \frac{1}{4(1+\cos x)} + C
\end{aligned}
$$

(4) 令 $t=\sqrt{x-2}$, 则 $x=t^2+2$, $\mathrm{d}x = 2t\,\mathrm{d}t$. 因此

$$
\begin{aligned}
\int \frac{x+1}{x\sqrt{x-2}}\,\mathrm{d}x &= 2\int \frac{t^2+3}{t^2+2}\,\mathrm{d}t = 2\int\left(1+\frac{1}{t^2+2}\right)\,\mathrm{d}t \\
&= 2\left[t+\frac{1}{\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right)\right]+C \\
&= 2\sqrt{x-2}+\sqrt{2}\arctan\left(\sqrt{\frac{x-2}{2}}\right)+C.
\end{aligned}
$$

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问题 5.2. 计算下列积分:

(1) $\displaystyle\int_a^{2a}\frac{\sqrt{x^2-a^2}}{x^4}\,\mathrm{d}x$;

(2) $\displaystyle\int_0^{\ln 2}\sqrt{1-e^{-2x}}\,\mathrm{d}x$;

(3) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+e^{\cos^2 x}}\,\mathrm{d}x$;

(4) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^4 x}{1+e^{-x}}\,\mathrm{d}x$.

Solutions.

(1) 为了去根号, 做如下换元 $x=\frac{a}{\cos\theta}$, 则 $\sqrt{x^2-a^2}=a\tan\theta$, $\mathrm{d}x = a\frac{\sin\theta}{\cos^2\theta}\,\mathrm{d}\theta$。当 $x=a$ 时 $\theta=0$, $x=2a$ 时 $\theta=\frac{\pi}{3}$. 因此

$$
\begin{aligned}
\int_a^{2a}\frac{\sqrt{x^2-a^2}}{x^4}\,\mathrm{d}x &= \int_0^{\frac{\pi}{3}}\frac{a\tan\theta}{\frac{a^4}{\cos^4\theta}}\cdot\frac{a\sin\theta}{\cos^2\theta}\,\mathrm{d}\theta = \frac{1}{a^2}\int_0^{\frac{\pi}{3}}\sin^2\theta\cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{3a^2}\sin^3\theta\Big|_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{8a^2}.
\end{aligned}
$$

(2) 做换元 $t=e^{-x}$, 则

$$
\int_0^{\ln 2}\sqrt{1-e^{-2x}}\,\mathrm{d}x = \int_1^{\frac{1}{2}}\sqrt{1-t^2}\cdot\left(-\frac{\mathrm{d}t}{t}\right) = \int_{\frac{1}{2}}^1\frac{\sqrt{1-t^2}}{t}\,\mathrm{d}t.
$$

再做三角换元, 设 $t=\sin\theta$ $(\mathrm{d}t=\cos\theta\,\mathrm{d}\theta)$, 积分上下限为 $\theta=\frac{\pi}{6}$ 到 $\theta=\frac{\pi}{2}$, 则积分化为:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sin\theta}\,\mathrm{d}\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\csc\theta-\sin\theta)\,\mathrm{d}\theta = \left(\ln\left|\tan\frac{\theta}{2}\right|+\cos\theta\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \ln(2+\sqrt{3})-\frac{\sqrt{3}}{2}.$$

(3) 做换元 $t=\cos^2 x$, 则

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+e^{\cos^2 x}}\,\mathrm{d}x &= \int_1^0 \frac{1}{1+e^t}\cdot(-\mathrm{d}t) = \int_0^1 \frac{1}{1+e^t}\,\mathrm{d}t \\ &= \int_0^1 \frac{e^{-t}\,\mathrm{d}t}{1+e^{-t}} = -\log(1+e^{-t})\Big|_0^1 = \log\frac{2e}{1+e}.\end{aligned}$$

(4) 设 $f(x)=\frac{\cos^4 x}{1+e^{-x}}$, 则 $f(-x)=\frac{\cos^4 x}{1+e^x}$, 且 $f(x)+f(-x)=\cos^4 x\cdot\frac{e^x+1}{e^x+1}=\cos^4 x$. 因此

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^4 x}{1+e^{-x}}\,\mathrm{d}x &= \int_{-\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(x)+f(-x)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x = \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}.\end{aligned}$$

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问题 5.3. (1) 设 $f(x)=\displaystyle\int_{\cos^2 x}^{2x^2} \frac{1}{\sqrt{1+t^2}}\,\mathrm{d}t$, 求 $f'(x)$.

(2) 已知 $\displaystyle\int_0^y e^{t^2}\,\mathrm{d}t + \int_0^{\sin x} \cos^2 t\,\mathrm{d}t = 0$, 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution. (1) 由微积分基本定理和链式法则, 得

$$f'(x) = \frac{4x}{\sqrt{1+4x^4}} + \frac{2\sin x\cos x}{\sqrt{1+\cos^4 x}}.$$

(2) 等式两边对 x 求导, 得

$$e^{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \cos^2(\sin x)\cos x = 0.$$

因此,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos^2(\sin x)\cos x}{e^{y^2}}.$$

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问题 5.4. (1) 求曲线 $\begin{cases} x=a\cos^3 t, \\ y=a\sin^3 t. \end{cases}\quad t\in[0,2\pi]$ 的弧长.

(2) 求曲线 $r=a(1+\cos\theta),\quad \theta\in[0,2\pi]$ 的弧长, 及其围成的区域的面积.

Solutions. (1) 曲线弧长为

$$\begin{aligned} L &= 4\int_0^{\frac{\pi}{2}} \sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t = 12a\int_0^{\frac{\pi}{2}} \sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}\,\mathrm{d}t \\ &= 12a\int_0^{\frac{\pi}{2}} \sin t\cos t\,\mathrm{d}t = 6a. \end{aligned}$$

(2) 曲线弧长为

$$L = 2\int_0^{\pi} \sqrt{r^2(\theta)+(r'(\theta))^2}\,\mathrm{d}\theta = 2\int_0^{\pi} \sqrt{2a^2(1+\cos\theta)}\,\mathrm{d}\theta = 4a\int_0^{\pi} \cos\frac{\theta}{2}\,\mathrm{d}\theta = 8a.$$

围成区域的面积为

$$S = 2\cdot\frac{1}{2}\int_0^{\pi} r^2(\theta)\,\mathrm{d}\theta = a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta = \frac{3}{2}\pi a^2.$$

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问题 5.5. 求 $y=\sin x, x\in[0,\pi]$ 绕 x 轴旋转一周得到的曲面的面积.

Solution. 旋转体的面积为

$$\begin{aligned} S &= 2\pi\int_0^{\pi} y(x)\sqrt{1+(y'(x))^2}\,\mathrm{d}x = 2\pi\int_0^{\pi}\sin x\sqrt{1+\cos^2 x}\,\mathrm{d}x \\ (\text{做换元 } t=\cos x)\quad &= 2\pi\int_{-1}^{1}\sqrt{1+t^2}\,\mathrm{d}t = 4\pi\int_0^1\sqrt{1+t^2}\,\mathrm{d}t \\ &= 4\pi\left[\frac{t}{2}\sqrt{1+t^2}+\frac{1}{2}\log(t+\sqrt{1+t^2})\right]\Bigg|_0^1 \\ &= 4\pi\left[\frac{\sqrt{2}}{2}+\frac{1}{2}\log(1+\sqrt{2})\right]. \end{aligned}$$

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问题 5.6. 设 $f\in C(\mathbb{R})$ 为 T-周期函数,

(1) 证明:$F(x)=\int_0^x f(t)\,\mathrm{d}t-\frac{x}{T}\int_0^T f(t)\,\mathrm{d}t$ 也是 T-周期函数.

(2) 证明:

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t.$$

(3) 求极限: $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x (t-[t])\,\mathrm{d}t$.

Proof. (1) 直接计算

$$F(x+T)-F(x)=\int_x^{x+T} f(t)\,\mathrm{d}t-\int_0^T f(t)\,\mathrm{d}t=0.$$

(2) 因为 $F(x)$ 是连续周期函数, 故 F 有界. (这是周期函数完全由它在一个周期 $[0,T]$ 上的值所决定, 而且有界闭区间上的连续函数有界.) 因此,

$$\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t + \frac{1}{x}F(x) \to \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t \qquad \text{as } x\to+\infty.$$

(3) 对 1-周期函数 $f(x)=x-[x]$ 用 (2) 的结论, 得

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^x (t-[t])\,\mathrm{d}t = \int_0^1 (t-[t])\,\mathrm{d}t = \frac{1}{2}.$$

□

问题 5.7. 设 f 是 $[0,a]$ 上的可积函数.

(1) 证明:

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,\mathrm{d}x = \frac{a}{2}.$$

(2) 计算:

$$\int_0^3 \frac{\log(1+x)}{\log(1+x)+\log(2-x)}\,\mathrm{d}x, \qquad \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\,\mathrm{d}x.$$

Proof. (1) 记等号左边的积分为 I, 做换元 $t=a-x$, 得

$$I = \int_0^a \frac{f(a-t)}{f(a-t)+f(t)}\,\mathrm{d}t = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,\mathrm{d}x.$$

故

$$2I = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,\mathrm{d}x + \int_0^a \frac{f(x)}{f(a-x)+f(x)}\,\mathrm{d}x = a.$$

(2) 利用 (1) 的结论直接得结果为 $\frac{3}{2}$ 和 $\frac{\pi}{4}$. □

问题 5.8. 设 f 在 $[0,+\infty)$ 上连续, 对任意 $a>0$, 证明:

$$\int_0^a \left(\int_0^x f(t)\,\mathrm{d}t\right)\mathrm{d}x = \int_0^a f(x)(a-x)\,\mathrm{d}x.$$

Proof. 记 $F(x)=\int_0^x f(t)\,\mathrm{d}t$, 由微积分基本定理, F 连续可导. 再由分部积分可得,

$$LHS = \int_0^a F(x)\,\mathrm{d}x = xF(x)\Big|_0^a - \int_0^a xF'(x)\,\mathrm{d}x = aF(a) - \int_0^a xf(x)\,\mathrm{d}x = \int_0^a f(x)(a-x)\,\mathrm{d}x.$$

□

问题 5.9. 设 $f\in C(\mathbb{R})$, 又设 $\varphi(x)=f(x)\int_0^x f(t)\,\mathrm{d}t$ 单调递减, 证明 $f\equiv 0$.

Proof. 注意到 $\varphi(x) = \left[\frac{1}{2}\left(\int_0^x f(t)\,\mathrm{d}t\right)^2\right]'$. 记 $F(x) = \frac{1}{2}\left(\int_0^x f(t)\,\mathrm{d}t\right)^2$, 因 φ 单调递减而且 $\varphi(0) = 0$, 故

$$F'(x)\begin{cases} \geq 0, & \text{当 } x < 0 \text{ 时}, \\ 0, & \text{当 } x = 0 \text{ 时}, \\ \leq 0, & \text{当 } x > 0 \text{ 时}, \end{cases}$$

因此, $F(x)$ 在 $x = 0$ 处取得最大值. 从而 $0 \leq F(x) \leq F(0) = 0$, 这说明了 $F \equiv 0$, 则 $\int_0^x f(t)\,\mathrm{d}t \equiv 0$. 因此

$$f(x) = \left(\int_0^x f(t)\,\mathrm{d}t\right)' \equiv 0.$$

□